The Schwarzschild Solution and an Introduction to Black Holes

PHYS 471: Introduction to Relativity and Cosmology

We have previously seen that Einstein's equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

These are second order partial differential equations for $g_{\mu\nu}$. For a particular choice of boundary conditions via the energy momentum tensor $T_{\mu\nu}$, we can find the spacetime metric that describes such a distribution. That's the basic game of General Relativity: give me a $T_{\mu\nu}$, and I will tell you the $g_{\mu\nu}$!

1 The First Solution: The Schwarzschild Metric

In the absence of any matter, the solution is the flat spacetime metric,

$$g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$$

because there is no curvature. If we want to know how spacetime acts around some mass M, however, we get something decidedly different. Clearly, the spacetime must curve, since any mass creates a gravitational field.

The first solution to Einstein's equations came only a few years after Einstein actually proposed them, and as luck would have it it *wasn't* by Einstein himself! Instead, a German physicist by the name of **Karl Schwarzschild** derived it, beating out Einstein because he closed the elevator door on him before he could get in. Actually, Schwarzschild was serving in the German army at the time of his derivation, stationed on the front lines of World War I! He asked himself what the most general, **spherically-symmetric metric would be for a static mass**. Using the stress-energy tensor for this case, with a mass M located at the origin of a spherical spacetime coordinate system, he found the following metric:

$$g_{\mu_n u} = \begin{pmatrix} 1 - \frac{2GM}{rc^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{2GM}{rc^2}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

which is alternatively expressed as the line element

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)\, dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \qquad (1)$$

2 Basic Properties of the Schwarzschild Solution

We can infer some interesting properties of the spacetime described by the Schwarzschild solution. First and foremost, it is **spherically symmetric**. This is clear because there is no coordinate dependence in the θ and $d\phi$ components, g_{22} and g_{33}. In fact, it looks exactly like the metric for a spherical space.

The difference arises in the t and r components, through the function

$$g_{00} = -g_{11}^{-1} = 1 - \frac{2GM}{rc^2}$$

These change as one moves radially from the mass. If we're very far away from it, the function in the metric approaches unity

$$\lim_{r\to\infty} \left(1 - \frac{2GM}{rc^2}\right) = 1$$

and so the metric reduces to

$$ds^2 = dt^2 - dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

It's the flat spacetime metric! We say the Schwarzschild solution is **asymptotically flat**, which is just a fancy way of saying that the effects of gravity vanish infinitely far away from the source. But of course, we already knew this from Newton!

But the more interesting behavior comes in the limit of small r. If we dial it down so $r \to 0$, the function in the components diverges!

$$\lim_{r\to 0} \left(1 - \frac{2GM}{rc^2}\right) \to \infty$$

We call this a **singularity**, and it tells us that something is very wrong with this solution at $r = 0$. Things in physics don't do infinity! This is a huge problem in general relativity, and we'll discuss it at much greater length shortly.

2.1 The Interior Schwarzschild Solution

The spacetime described by the Schwarzschild metric applies to *any* static, spherically symmetric mass distribution. This includes stars, planets, and so forth (we have to ignore their rotation for the time being, but that's actually a fairly reasonable thing to do in this case). Clearly, the Sun and Earth do not have singularities at their core, where spacetime curvature becomes infinite! What to do?

It turns out that one can derive a metric for the interior of mass distributions that gets away from this unfortunate behavior. Let's assume the mass is a spherical distribution of radius R and density $\rho = \frac{M}{V(R)}$, where $V(R) = \frac{4}{3}\pi R^3$. The amount of mass contained within a radius $r < R$ is then

$$M(r) = V(r)\rho = \frac{r^3}{R^3}M$$

If we substitute this back into the mass term in the Schwarzschild metric, we find

$$\frac{2GM(r)}{rc^2} = \frac{2GMr^2}{R^3c^2}$$

and so the entire metric is

$$ds^2 = \left(1 - \frac{2GMr^2}{R^3c^2}\right) dt^2 - \frac{dr^2}{1 - \frac{2GMr^2}{R^3c^2}} - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

And presto! The singularity is gone! It's easy to see that when $r \to 0$, the metric components go to 1, and thus spacetime is flat at the center of the mass distribution. Also, note that when $r = R$, the solution matches the usual Schwarzschild solution.

2.2 Gravitational Time Dilation

Special Relativity tells us that moving clocks run slower than those at rest, so observers in different frames of reference will not agree on the passage of time. Specifically, an observer will see a proper time $d\tau^2 = \sqrt{1 - \frac{v^2}{c^2}dt^2}$, which is slower than the coordinate time interval dt in the rest frame. This is special relativistic time dilation.

A similar effect arises in General Relativity, which can be seen from the Schwarzschild metric. If we consider only the dt component of the metric, then we have

$$ds^2 = d\tau^2 = \left(1 - \frac{2GM}{rc^2}\right) dt^2 \quad \Longrightarrow \quad d\tau = \sqrt{1 - \frac{2GM}{rc^2}}dt$$

This tells us that an observer positioned a distance r from the center of the mass M experiences a passage of time that is **less** than that of an observer at $r \to \infty$.

Gravitational fields cause clocks to run slowly!

This effect is pretty small for the Earth. The difference between a clock on the surface ($r_1 \approx 6000$ km) and a clock on a satellite orbiting 20 000 km above the surface ($r_2 = 26\ 000$ km) is less than 0.0000000001 seconds! This effect becomes much more important when the mass is *extremely* large.

As you get sufficiently far away from the mass, your proper time and the coordinate time will agree (take $r \to \infty$ and check for yourself!).

2.3 Gravitational Redshift and Blueshift

In a similar fashion to the Doppler effect, the wavelength (and frequency) of a beam of light emitted from or heading toward a gravitational source can shift. Let's consider frequency, which goes as $\nu \sim \frac{1}{d\tau}$. Using the result for gravitational time dilation, we can surmise the frequency ratio for a beam of light at two different radii from a mass is

$$\frac{\nu_2}{\nu_1} = \frac{d\tau_1}{d\tau_2} = \sqrt{\frac{1 - \frac{2GM}{r_1 c^2}}{1 - \frac{2GM}{r_2 c^2}}}$$

Specifically, what happens to a ray of light that is observed far from the source? If we take $r_1 = R$ (radius of the source) and $r_2 \to 0$, then we see

$$\frac{\nu_\infty}{\nu_1} = \sqrt{1 - \frac{2GM}{Rc^2}} \quad \Longrightarrow \quad \nu_\infty = \nu_1 \sqrt{1 - \frac{2GM}{Rc^2}}$$

Since the term under the square root is less than 1, this means that frequency observed at infinity will be **less than** the frequency of the original beam. That is, the light is **gravitationally redshifted** as it moves away from the source. If we reverse the direction of travel so that the light is heading toward the mass, it follows that the beam will be **blueshifted**.

2.4 Proper Length

If the curvature affects the passage of time, it will also affect the definition of length. For this, let's consider a radial path out of a mass M. It's proper length will be defined as

$$ds^2 = \frac{dr^2}{1 - \frac{2GM}{rc^2}} \quad \Longrightarrow \quad ds = \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}}$$

This seems to be the opposite effect of gravitational time dilation! Lengths apparently become **longer** in a gravitational field. Integrating this between two coordinate points

r_1 and r_2, we find

$$\Delta s = \int_{r_1}^{r_2} ds = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{2GM}{rc^2}}}$$

You can easily convince yourself that this is true by doing the integral above. But how do we interpret this? A straightforward way to think about it is with our sphere analogy. As you move along the surface of the Earth, you travel along an arc, and so the distance you travel is the arc length. This is longer than a straight line (chord) that connects the two points. So, curvature increases the length between points, and the greater the degree of curvature, the greater the length.

A more formal explanation is that the curvature is actually stretching the space between two points. We'll see this in a little bit when we visualize the Schwarzschild solution through what's called an *embedding diagram*.

3 Embedding Diagrams

We've all seen the depiction of a black hole spacetime as the "bowling ball on a sheet", that is, a funnel shape. This is an example of what is known as an **embedding diagram**. Since the space that is curved is three-dimensional, we can't really "see" it tunneling down into some other dimension. But we *can* get a sense of what it looks like by studying the radial behaviour of a function $z(r)$ of some hypothetical "height". We can imagine $z(r)$ as the "topographical" map of the spacetime curvature.

To construct an embedding diagram and find the embedding function $z(r)$, we consider the metric at a fixed point in time ($dt = 0$) and for a specific orientation by setting $d\theta = 0$, and choosing $\theta = \frac{\pi}{2}$,

$$ds^2 = \frac{dr^2}{1 - \frac{2GM}{rc^2}} + r^2 d\phi^2$$

This is effectively a two-dimensional surface. To add "depth", we *embedding* this surface in a cylindrical geometry:

$$ds^2_{\text{cyl}} = dr^2 + r^2 d\phi^2 + dz^2$$

The shape of the curvature in the z-direction in an embedding diagram depends on the radial coordinate, so we replace $dz \to \frac{dz}{dr}\,dr$:

$$ds^2_{\text{cyl}} = dr^2 + r^2 d\phi^2 + \left(\frac{dz}{dr}\right)^2 dr^2$$

$$\implies \quad ds^2_{\text{cyl}} = dr^2 \left(1 + \left(\frac{dz}{dr}\right)^2\right) + r^2 d\phi^2$$

Finally, we equate ds^2 and ds^2_{cyl}:

$$\begin{aligned} ds^2 &= ds^2_{\text{cyl}} \\ \frac{dr^2}{1 - \frac{2GM}{rc^2}} &= \left(1 + \left(\frac{dz}{dr}\right)^2\right) dr^2 \end{aligned}$$

and the $d\phi^2$ terms cancel. We now have a differential equation for $z(r)$ in terms of the metric function:

$$\frac{1}{1 - \frac{2GM}{rc^2}} = \left(1 + \left(\frac{dz}{dr}\right)^2\right)$$

which gives us

$$\frac{dz}{dr} = \frac{1}{1 - \frac{2GM}{rc^2}} - 1$$

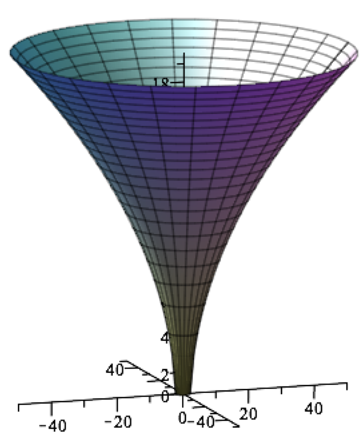
The embedding function $z(r)$ is therefore

$$z(r) = \int dz = \int_0^r \left(\frac{1}{1 - \frac{2GM}{r'c^2}} - 1\right) dr'$$

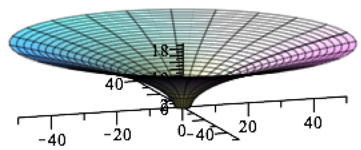
In fact, for *any* metric whose rr component is specified by some function $g_{rr} = -f(r)^{-1}$, the embedding function is

$$z(r) = \int_0^r \left(f(r')^{-1} - 1\right) dr'$$

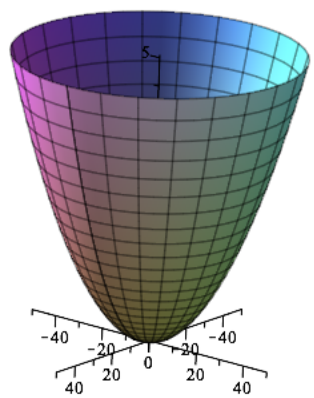
What does this look like for the Schwarzschild metric? We can use Maple (!!) to evaluate the integral, and then plot it. We find the expected funnel shape:



This doesn't quite look like what we expect, because the axes aren't fixed to the same scale. If we do that, we do get the familiar image!



We can also look at the interior solution to the Schwarzschild solution, and confirm that the curvature smoothly goes to zero at the center:



4 The Curvature Behavior of the Schwarzschild Solution

So far, we haven't talked about the Christoffel symbols, Riemann tensor, or Ricci tensor and scalar of the Schwarzschild solution! What's wrong with us? We can easily calculate these things using Maple (hand calculations are sooooo first half of the semester). We find that the Schwarzschild solution has 9 non-vanishing Christoffel symbols, and $\frac{4^2(4^2-1)}{12} = 20$ Riemann components. Since these are non-zero, we know the spacetime is curved. What is interesting, however, is the Ricci tensor and Ricci scalar are 0! This effectively means that Einstein's equations, in the absence of a cosmological constant, are identically zero as well...

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \quad \Longrightarrow \quad T_{\mu\nu} = 0$$

Huh? How can the energy-momentum tensor be 0 if there is something there to create it (the black hole itself!). This is a bit of an enigma in General Relativity, and dates back to the original derivation of the solution itself. There really isn't "no source", but rather it's a point source at $r = 0$:

$$T_{00} = \rho\, \delta(r)$$

This misunderstanding has resulted in a historical misrepresentation of the Schwarzschild solution as a "vacuum solution" (which it isn't), as opposed to a "solution in vacuum" (which it is). This subtlety is probably more that you need to know at this stage, so we'll leave it at that!

5 The Schwarzschild Radius

The most interesting things about black holes happen at the radius at which the metric function vanishes:

$$1 - \frac{2GM}{rc^2} = 0 \quad \Longrightarrow \quad r = R_S = \frac{2GM}{c^2}$$

At this point, the time component g_{00} of the metric vanishes, and the radial component g_{11} diverges! This is called the **Schwarzschild radius**, and some very, very, *VERY* strange physics happens here.

Typical objects – planets, stars, galaxies, you – have Schwarzschild radii that are much smaller than their actual size. The Schwarzschild radius of the Sun, for example, is

$$\text{Sun} \quad \longrightarrow \quad R_S = \frac{2(6.67 \times 10^{-11})(2 \times 10^{30})}{(3 \times 10^8)^2} \approx 3000 \text{ m}$$

The Sun is clearly larger than 3 km! Likewise, the Schwarzschild radius of the Earth is

$$\text{Earth} \quad \longrightarrow \quad R_S = \frac{2(6.67 \times 10^{-11})(6 \times 10^{24})}{(3 \times 10^8)^2} \approx 0.009 \text{ m}$$

The Earth is clearly larger than one millimetre! Just for fun, your Schwarzschild radius is about

$$\text{You} \quad \longrightarrow \quad R_S \approx 10^{-25} \text{ m}$$

The Schwarzschild radius is effectively meaningless for mass distributions which extend far beyond this distance. But what if an object were so dense that it was entirely contained within the Schwarzschild radius??? In that case, you would have a...

6 Black Holes

Black hole? What's that? And why is it black? These objects have been hypothesized for well over 200 years now, but no one could make any sense of them until Schwarzschild proposed his solution to Einstein's equations. The earliest appearance of a black hole in the scientific literature came from John Michell (a geologist!) in 1783. He asked himself how big an object would have to be to have an escape velocity equal to the speed of light. A simple Newtonian calculation shows that by conservation of energy,

$$K = U \quad \Longrightarrow \quad \frac{1}{2}mc^2 = \frac{GmM}{R}$$

$$\Longrightarrow \quad R = \frac{2GM}{c^2}$$

This is the Schwarzschild radius, derived 132 years before the Schwarzschild radius wasactually derived!! Michell termed these objects **dark stars**, since the light they emit would "fall back" to the surface and they would therefore not be detectable. Since no one had any clue about relativity theory or spacetime a this time, it lingered as nothing more than a curiosity.

Black holes can be formed in one of three ways, which determines the possible mass range in which they can exist. These include:

- **Astrophysical black holes:** These objects result from the collapse of aging stellar material due to the inward gravitational pressure overpowering the outward radiation and nuclear pressure. In the 1930s, physicist Subrahmanyan Chandrasekhar noted that dying stars of mass $M_{\text{star}} \leq 1.4 M_\odot$ will settle down to a white dwarf. This is known as the **Chandrasekhar limit**. Those of mass exceeding this value *will* collapse further as the gravitational pressure overwhelms the nuclear pressure, resulting in the mass compacting interior to the object's Schwarzschild radius.

- **Primodrial black holes (PBHs):** In the high energy chaos following the big bang, violent quantum fluctuations could have created sufficient densities to form black holes. This is effectively done by creating an amount of energy $E = Mc^2$ within a region of space $R_S = 2GM/c^2$. There is debate as to whether this needs to be completely spherically symmetric, or if instead the matter should be contained within an *average* radius $\langle R \rangle \sim R_S$ (known as the **hoop conjecture**, an unproven assumption put forth by Kip Thorne... he gets around!). There is no specific limit to the masses of PBHs (well, they can't exceed the mass of the Universe, so there's that), but it is believed that the range could be from $10^{-12}\ M_\odot$ through $10^{10}\ M_\odot$. The latter are known as **supermassive black holes**, and are suspected to sit at the center of most galaxies. These began life as PBHs and slowly accreted matter, resulting in "seeds" that eventually formed galaxies.

- **Quantum black holes (QBHs):** When two high energy particles collide and create a sufficient high energy density in a region the size of the corresponding Schwarzschild radius (hoop conjecture again!), the result could be a super-mini-sized black hole. These aren't likely to form unless the theory of gravity permits it, which General Relativity does not. Alternative theories of gravitation, however, do. We'll discuss these later in the semester!

Fast forward to 1915, and we now see the Schwarzschild radius R_S defines a coordinate singularity in the spacetime around a static mass distribution. Plugging in this value for the gravitational time dilation and redshift formulae, we notice the following:

- Time dilation at R_S: $\quad d\tau_{R_S} = \sqrt{1 - \frac{2GM}{R_S c^2}}\, dt = 0$

- Redshift from R_S: $\quad \nu_\infty = \nu_{R_S}\sqrt{1 - \frac{2GM}{R_S c^2}} = 0$

This is all bizarre. Clocks apparently stop at R_S and light emitted from R_S is observed to have no frequency! It's almost like the light didn't make it out from R_S in the first place.

All this points to the idea that light just ain't gonna make it away from the Schwarzschild radius. The dark stars from 1783 really are dark – we can't see anything past the Schwarzschild radius. So, in nautical parlance, we'll call this a **horizon**. And in relativity parlance, we'll say we can't see any events that occur beyond this point: **the event horizon**!

7 Is the Event Horizon Singularity Real?

Using Maple, we have the awesome power to calculate all curvature components! Since these are all functions of the coordinates, in particular r, we can see if there are equally weird things that happen to them at the Schwarzschild raidus $r = R_S$. Although some of the Riemann components do diverge at this point, some also don't. If only there were *one* particular value of the Riemann tensor, or a relative thereof, that we could look to for insight.

In fact, there is! The **Kretschmann scalar** is defined as $\mathcal{K} = R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$, and it is a number that combines all the Riemann tensor components into a single value. Since it's also Lorentz invariant, then whatever happens to it must happen in *all* frames of reference and coordinate systems. So, if we ask Maple to calculate it (or do it ourselves, just to be sure!), we find that it has the value

$$\mathcal{K} = \frac{G^2 M^2}{r^6 c^4}$$

which clearly *does not* diverge at $r = R_S$. In fact, it is quite finite there, and only diverges when $r = 0$. So, **the** singularity is real and is something we need to deal with, but the event horizon singularity is **not real**. So what is it?

8 A Student Walks into a Black Hole and...

OK, there's no joke that I know of that starts like that. But let's examine what happens to an observer who falls into a black hole, and pay particular attention to what happens at R_S. If we're sitting out at infinity and push them toward the black hole, they will fall along a geodesic and, according to us, eventually freeze at the horizon. But what do they experience? We need to transform ourselves into the frame of the infalling observer. In order to do so, however, we'll need to understand orbits around – and into – black holes. Stay tuned to the next handout!