Schwarzschild Orbits – Particles and Photons
PHYS 471

We have been studying the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2 d\theta^2 - r^2 \sin^2\theta \; d\phi^2$$

which is the only static, spherically-symmetric solution to Einstein's equations for a gravitating point mass. Let's now assume that $c = 1$, so the metric is

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2GM}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta \; d\phi^2$$

In the following pages, we'll examine what the orbits of both particles (matter) and photons (light) look like in this spacetime geometry. More importantly, we'll study orbital motion around black holes...

1 Constraints for Orbital Motion

As we have previously discussed, freefall motion in any spacetime is described by the geodesic equations for each coordinate. These are calculated as

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\ \mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{1}$$

or from the alternative form

$$\frac{d}{d\tau}\left(g_{\alpha\beta}\frac{dx^\beta}{d\tau}\right) - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\mu}{d\tau} = 0 \tag{2}$$

The second version will prove easier to use when discussing orbits, so we'll stick with that. It also has the advantage that it by-passes calculation of those pesky Christoffel symbols!

The geodesic equations tell us about the shortest *path* an object will take, but it actually doesn't constrain the dynamics of the object. To date, we were just putting those in by hand (*e.g.* constant velocity in one direction, no velocity in another, and so forth). We thus need on more condition to constrain the orbital motion. Since one of the biggest things in physics is **conservation of energy**, it stands to reason that this is probably a good law to obey.

In classical mechanics, the conservation equation is $E_{\text{tot}} = K + U$, where K is the kinetic energy and U the potential. For a gravitational system, this is

$$E_{\text{tot}} = \frac{1}{2}mv^2 - \frac{GmM}{r}$$

If we divide through by m, the result is independent of the "observer", and makes it kinda like an invariant quantity:

$$\mathcal{E}_{\text{tot}} = \frac{E_{\text{tot}}}{m} = \frac{1}{2}v^2 - \frac{GM}{r}$$

This is an expression for the energy per unit mass, and from this we can get things like escape velocity and orbital periods. Remember the mantra: "orbits care not for the orbiter, only the orbited!"

But we're studying relativity, not Newtonian physics! We need to update those terms accordingly, while keeping the same mantra (orbited matters, not orbiter). Relativistic energy conservation is expressed by the Lorentz invariant magnitude of the four-momemtum (squared),

$$m^2 = p^\mu p_\mu \quad \Longrightarrow \quad m^2 = (\gamma m)^2 - (\gamma|\vec{p}|)^2$$

If we divide *this* through by m as we did in the Newtonian case, we get an expression for the four-velocity (squared),

$$1 = \gamma^2 - \gamma^2|\vec{v}|^2$$

which is true for any velocity (check it!).

Written as a vector, in a rest frame the four-momentum is

$$p^\mu = (m, 0, 0, 0)$$

and so it follows that in the rest frame, the four-velocity is

$$u^\mu = (1, 0, 0, 0)$$

That is, the magnitude of the four-velocity for *any* particle is $u^2 = u^\mu u_\mu = 1$. In terms of the metric, we can write this as

$$u^2 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 1$$

This is called a **timelike velocity**, because it's constrained to lie within the light cone.

What about conservation of relativistic energy for photons? We have seen in several instances that the four-momentum vector for a photon is $w^\mu = (\omega, 0, 0, \omega)$, which is a **null vector**, $w^\mu w_\mu = \omega^2 - \omega^2 = 0$. So, voilà! (that's French, eh) Energy conservation for a photon must be determined by the condition

$$w^2 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

As you probably already said to yourself, this is called a lightlight velocity, because, well, it describes light! Moreover, it is a velocity constrained to lie *on* the lightcone, which itself is a null vector.

An added bonus of this description is that these conditions are particularly important in our understanding of what happens to objects that fall into black holes, because as mentioned, they are evaluated *in the rest frame of the object itself.* We already know what happens to a person falling toward a black hole as viewed from our perspective. These will help show what happens from *their* perspective!

2 Particle Orbits

Let's first consider the orbit of a massive particle. In this case, the four-velocity magnitude is

$$1 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

or

$$1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + g_{\theta\theta}\left(\frac{d\theta}{d\tau}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2$$

Now for conservation of energy. Using the second form of the geodesic equation (2) for the time coordinate, we get

$$\frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) = 0 \tag{3}$$

Since this derivative is zero, we know that the quantity in the parentheses must be constant:

$$g_{tt}\frac{dt}{d\tau} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} = e \tag{4}$$

From the unit perspective, we should recognize this as a type of **rest energy per unit mass**.

The radial geodesic equation will give us something messy, so we'll skip to the θ and ϕ equations:

$$\begin{aligned}\frac{d}{d\tau}\left(g_{\theta\theta}\frac{d\theta}{d\tau}\right) - \frac{1}{2}(\partial_\theta g_{\phi\phi})\left(\frac{d\phi}{d\tau}\right)^2 &= 0 \\ \frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right) &= 2r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2\end{aligned}$$

and

$$\frac{d}{d\tau}\left(g_{\phi\phi}\frac{d\phi}{d\tau}\right) = 0 \quad\Longrightarrow\quad r^2\sin^2\theta\frac{d\phi}{d\tau} = \ell = \text{const.} \tag{5}$$

where ℓ is the angular momentum per unit mass of the orbiting body.

If we restrict ourselves to the equatorial plane $\theta = \frac{\pi}{2}$, these geodesic equations reduce to

$$\frac{d}{d\tau}\left(r^2 \frac{d\theta}{d\tau}\right) = 0 \quad \Longrightarrow \quad r^2 \frac{d\theta}{d\tau} = \text{const}$$

$$r^2 \frac{d\phi}{d\tau} = \ell$$

The symmetry between the angular equations tells us that it doesn't matter which plane we choose to set our orbit. Whichever one it is, the orbit will remain in that plane indefinitely (from the condition $\frac{d\theta}{d\tau} = 0$).

Given these constraints, the energy equation derived from the four-velocity condition becomes

$$1 = \left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2$$

Substituting in our above expressions (4) for e and (5) for ℓ, and doing a little algebra, gives

$$\mathcal{E} = \frac{1}{2}\left(e^2 - 1\right) = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3} \tag{6}$$

The quantity $\mathcal{E}$ is the **total relativistic energy per unit mass** and is **conserved**. The other terms are

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 \quad \longrightarrow \quad \text{Radial kinetic term}$$

$$-\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3} \quad \longrightarrow \quad \text{Effective potential term}$$

We'll call the second one the **effective relativistic potential**

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3}$$

which differs from the Newtonian potential by the addition of the term $-\frac{GM\ell^2}{r^3}$. Figure 1 shows the relativistic potential.

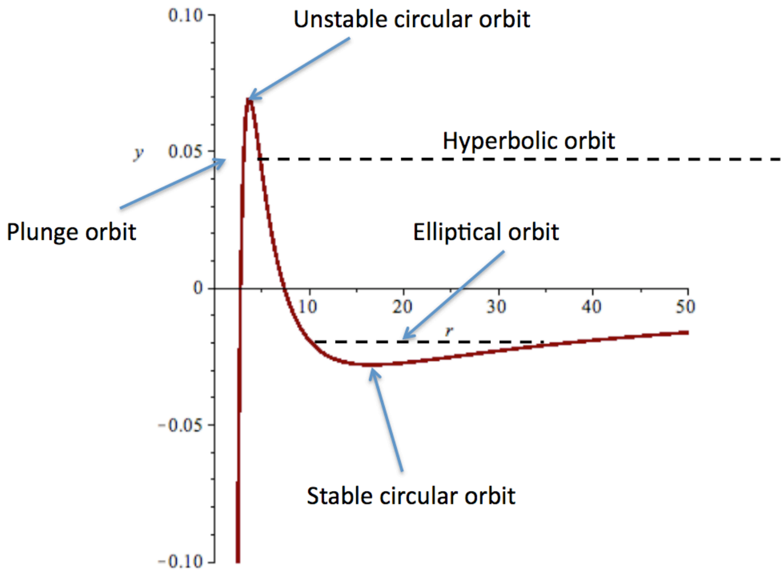


Figure 1: Effective potential $V_{\text{eff}}(r)$ for particle orbits. The Newtonian effective potential diverges to $V_{\text{Newt}} \to \infty$ as $r \to 0$, whereas the relativistic potential goes to $V_{\text{eff}} \to -\infty$

There are four types of possible orbits that can result in this potential, depending on the total energy of the particle (detailed in Figure 1):

2.1 Hyperbolic orbits

when $\mathcal{E} > 0$, the object will be unbound and if it comes from "infinity," it will go back to "infinity."

2.2 Elliptical orbits

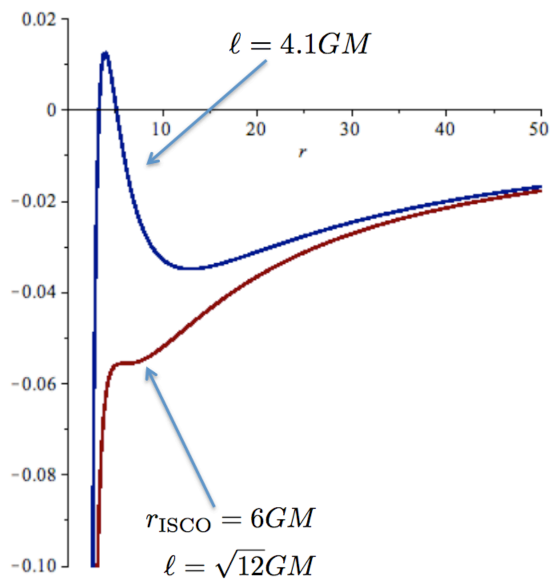
when $\mathcal{E} < 0$, the object will be in a bound orbit with a perihelion (closest approach to the orbited mass M) and an aphelion (further approach).

2.3 Circular orbits

In two special cases, the orbits will be exactly circular. These occur when the potential is minimum, $\frac{dV_{\text{eff}}(r)}{dr} = 0$. Differentiating our potential gives two solutions,

$$\frac{dV_{\text{eff}}(r)}{dr} = 0 \quad \Longrightarrow \quad r_{\pm} = \frac{\ell^2}{2GM}\left(1 \pm \sqrt{1 - \frac{12G^2M^2}{\ell^2}}\right)$$

These correspond to the maximum (r_-) and minumum (r_+) points in Figure 1. The former is an **unstable circular orbit**, since perturbations to either side will result in either an escape, or a plunge into the singularity. The latter is a **stable circular orbit**.



Note that when the quantity under the radical vanishes, we get $\ell = \sqrt{12}GM$, and the corresponding radius is $r_{\rm ISCO} = \frac{\ell^2}{2GM} = 6GM$. This is called the **innermost stable circular orbit**. No orbits with $r < r_{\rm ISCO}$ can exist, and this leads to...

2.4 Plunge Orbits

... an orbit in which the object simply falls into the orbiting mass! If it wanders inside the ISCO, the curvature is too great to establish any kind of stable orbit, and the mass eventually crashes into the mass it's orbiting (or into the event horizon, if it's a black hole).

3 Photon Orbits

If the object approaching the mass M is a photon, then the equations change slightly. While we can still use the geodesic equations as outlined before (with the same definitions for e and ℓ), we account for the massless nature of the photon by using a **lightlike velocity 4-vector**

$$0 = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

This gives the new energy conservation equation of the form

$$0 = \left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\phi}{d\tau}\right)^2$$

Rearranging this, we obtain

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) \quad , \quad b^2 = \frac{e^2}{\ell^2}$$

As before, we see that this equation defines a new effective potential for the photon,

$$W_{\text{eff}} = \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)$$

whose shape is shown in Figure 2.

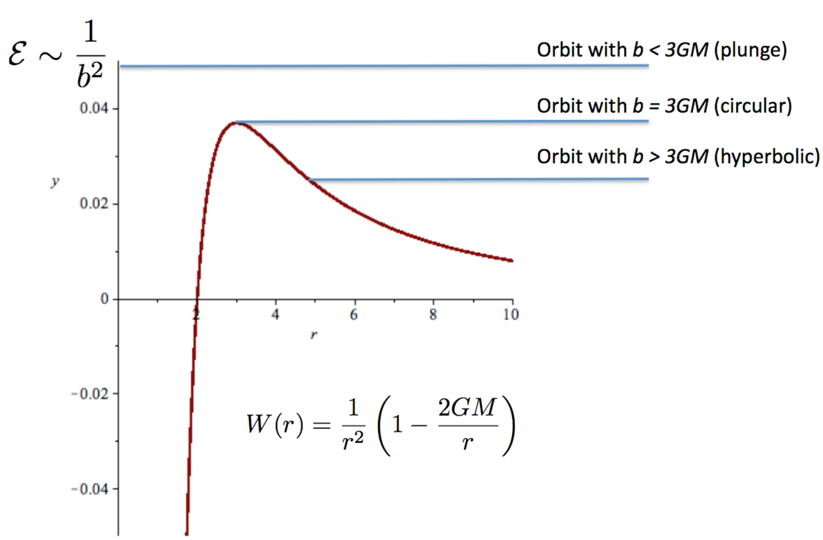


Figure 2: Effective potential $W_{\text{eff}}(r)$ for photon orbits, showing three orbital conditions depending on the impact parameter b.

It has a maximum when

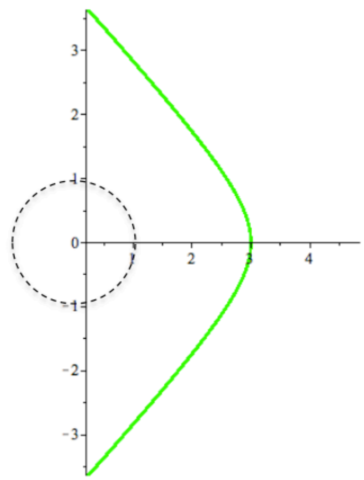
$$\frac{dW_{\text{eff}}}{dr} = 0 \implies r = 3GM = 1.5r_s$$

This is called the **photosphere**, where photons orbit M in a circular fashion! Note when $r = r_{\text{photosphere}} = 3GM$, the potential is $W(r_{\text{photosphere}}) = \frac{1}{27G^2M^2}$.

It turns out the quantity b has units of *length*, and can be shown to be the distance of closest approach of the photon to M. This is called the **impact parameter**, and its value serves to define one of **three possible orbits** described below:

3.1 Hyperbolic /scattering orbits

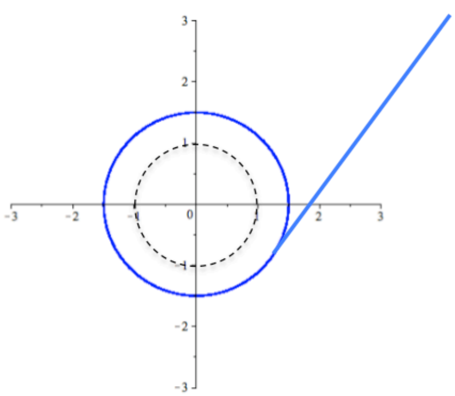
Photons with $b > 3GM$ will be deflected by the curvature induced by M, but will escape to infinity. The dotted line is the Schwarzschild radius.



Orbit with *b* > *3GM* (hyperbolic)

3.2 Circular orbits

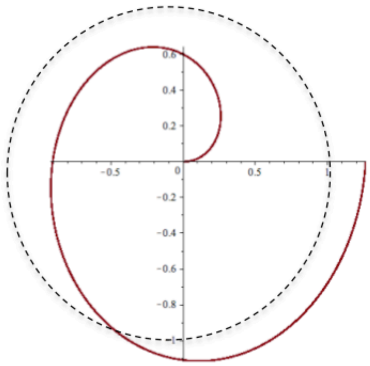
Photons having impact parameter $b = 3GM$ will orbit M indefinitely.



Orbit with *b* = *3GM* (circular)

3.3 Plunge orbits

Photons having $b < 3GM$ will spiral into M.



Orbit with *b < 3GM* (plunge)

4 Solar System Tests of General Relativity

The orbits described above provide a good testbed for general relativity, using the objects in our immediate area of the universe. Specifically, we can determine how the orbit is affected by examining motion in the ϕ direction. This manifests itself in both cases as **advance of perihelion** (massive particles) and **deflection of starlight** (photons).

The gist of the approach is to find a function $\phi(r)$ that determines the angular orbital coordinate as a function of distance from M. Since the curvature gets stronger for decreasing r, this will impact the orbital motion by changing the path of the geodesic. We use the energy conservation equation $\mathcal{E}$ with the general substitution

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{dr}{d\phi}\frac{\ell}{r^2}$$

and rearrange to solve for $r(\phi)$.

4.1 Advance of Perihelion

In the case of massive particles, we get

$$\frac{dr}{d\phi} = \frac{r^2}{\ell}\sqrt{e^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)}$$

Both solutions ($\pm$) are equally valid, but the sign just determines the direction of the orbit. So, we'll keep it positive. Now we ideally want to find $\phi(r)$, not $r(\phi)$, we obtain the differentials

$$d\phi = \frac{\ell}{r^2}\sqrt{e^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)}^{\,-1}\, dr$$

Integrating the dr part over the perihelion and aphelion gives

$$\Delta\phi = \int_{r_{\rm peri}}^{r_{\rm ap}} \frac{\ell}{r^2} \frac{dr}{\sqrt{e^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)}}$$

The quantity $\Delta\phi$ is the numbers of radians swept out in a complete orbit, when the starting point and end point are at the same location in the ellipse. Orbits that do not precess have $\Delta\phi = 2\pi$, while orbits that do precess have $\Delta\phi > 2\pi$. We define the **advance of perihelion** to be this excess, $\delta\phi_{\prec} = \Delta\phi - 2\pi$.

As mentioned in class, the above integral is difficult to solve exactly since it is an elliptical function. In fact, one must resort to numerical methods to solve it. When the curvature is particularly strong (*i.e.* for orbits near or inside $R_{\rm ISCO}$), the resulting calculation needs to be extremely precise. That means heavy computation time.

In the solar system, however, the curvature is pretty slight. This means $2GM_{\rm sun} \ll 1$, and so we can make some approximations by doing series expansions in this quantity. The upshot is a nice expression of the form

$$\delta\phi_{\rm prec} = \frac{6\pi G^2 M^2}{c^2 \ell^2}$$

where c has been re-introduced for computational ease. Using Kepler's law of equal areas, $\ell^2 = GMa(1-\epsilon)$ (where a is the semi-major axis of the orbit and ϵ the eccentricity), can manipulate the above expression to find

$$\delta\phi_{\rm prec} = \frac{6\pi G^2 M^2}{c^2 a(1-\epsilon^2)}$$

Things to note:

- The precessional advance will be large when a is small
- The precessional advance will be large when M is large
- The precessional advance will be large when $\epsilon > 0$

So, planets close to Sun with large eccentricities will show a large precession. This, of course, exactly describes Mercury, whose values are $a = 5.7 \times 10^{10}$ m and $\epsilon = 0.206$. Plugging in these values gives $\delta\phi_{\rm prec} = 5.12 \times 10^{-7}$ rad, or $\delta\phi_{\rm prec} = 2.9 \times 10^{-5}$ degrees. So, every time Mercury orbits, it advances by this tiny amount. Since Mercury orbits the Sun four times a year, we can rework this quantity into a more historic unit or "arcseconds per century". This is

$$\delta\phi_{\rm prec} = 0.012 \text{ deg/century} = 42.2 \text{ arcsec/century}$$

This is *precisely* the value observed! Einstein's theory predicted this hitherto explained phenomenon to an incredibly high accuracy.

4.2 Deflection of Starlight

An incoming photon will deflect according to its impact parameter b. In a similar fashion to the advance of perihelion, we can derive an estimate for the deflection angle $\Delta\phi_\star$. We set $\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{dr}{d\phi}\frac{\ell}{r^2}$, and this time substitute this into the null energy condition equation:

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\phi}\right)^2 \frac{\ell^2}{r^4} + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)$$

$$\implies \quad \frac{dr}{d\phi} = \frac{r^2}{b}\sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)}$$

and so

$$d\phi = \frac{b}{r^2}\frac{dr}{\sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)}}$$

Integrating this from the impact parameter (closest approach) to $r \to \infty$ gives half the angle, so we multiply by two to obtain

$$\Delta\phi_\star = 2b\int_b^\infty \frac{1}{r^2}\frac{dr}{\sqrt{1 - \frac{b^2}{r^2}\left(1 - \frac{2GM}{r}\right)}}$$

Again, this is an icky-yucky integral that even the best of computer algebra packages (ahem... Maple) have trouble evaluating. Making some series estimates for small $r << 2GM$ as before, however, and we can cut to the chase and obtain the approximate formula

$$\Delta\phi_\star \approx \frac{4GM}{bc^2}$$

where again the speed of light has been introduced for calculational convenience. For typical starlight passing close to the Sun ($b \sim R_{\text{Sun}} = 7 \times 10^8$ m), we find

$$\Delta\phi_\star \approx 8.5 \times 10^{-6} \text{ rad} = 1.75 \text{ arcsec}$$

This again precisely matches the observed deflection during a solar eclipse.

In the solar system, we can