The Geodesic Equation
PHYS 471: Intro to Relativity and Cosmology

1 Straight lines are defined by $\vec{a} = 0$

As we all know, in the absence of an external force, objects will move in straight lines. If you don't know this, please withdraw from this course and revisit PHYS 101! But what *is* a straight line? A suitable definition might be "a straight line is the shortest distance between two points", which certainly fits the bill. In fact, it's pretty much the correct definition.

Mathematically, we can think of a straight line as being the path of an object whose acceleration is $a = 0$. Fancying up this statement, we could say that if an object's position is described by the vector $\vec{r}(t)$ at some time t, then

$$\frac{d^2\vec{r}(t)}{dt^2} = 0$$

ensures that it follows a straight line path. So far, so good! If it works in Newtonian mechanics, then all we need to do is rev it up to relativistic mechanics....

2 Straight lines are defined by $a^\mu = 0$

And there we have it! To go from the Newtonian to the relativistic world, we replace spatial vectors with spacetime vectors. This has added baggage to it as well, because we are no longer taking simple time derivatives, but rather proper time derivatives. So, if the position of an object is defined to be x^μ, then the straight line must be defined by the condition

$$\frac{d^2x^\mu}{d\tau^2} = 0$$

where τ is the proper time. This works well if we have a cartesian-like coordinate system, but by now you realize that if you introduce a new coordinate system, or (ack!) curvature, you'll need to include Christoffel symbols. So...

3 The Geodesic Equation

We start with a position four-vector $\mathbf{x}$, where $\mathbf{x} = x^\mu \hat{e}_\mu$ and ask how this varies with proper time. According to the product rule, the result is

$$\frac{d\mathbf{x}}{d\tau} = \frac{dx^\mu}{d\tau}\hat{e}_\mu + x^\mu \frac{d\hat{e}_\mu}{d\tau}$$

As with the definition of the covariant derivative, we see that we need to take into account how the basis vectors $\hat{e}_\mu$ change over (proper) time, in addition to the vector component itself. But how does one take a proper time derivative of $\hat{e}_\mu$? Again, remember your calculus trickery! We can use a chain-eseque type rule to write

$$\frac{d\hat{e}_\mu}{d\tau} = \frac{\partial \hat{e}_\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}$$

That twitch in your eye and sense of impending dread you feel means you have been well-conditioned to GR, and recognize that the Christoffel symbol makes an appearance! That is,

$$\frac{\partial \hat{e}_\mu}{\partial x^\nu} = \Gamma^\alpha{}_{\mu\nu}\hat{e}_\alpha$$

and so

$$\frac{d\mathbf{x}}{d\tau} = \frac{dx^\mu}{d\tau}\hat{e}_\mu + x^\mu \Gamma^\alpha{}_{\mu\nu}\hat{e}_\alpha \frac{dx^\nu}{d\tau}$$

Since the μ index in the first term is a summed one (dummy!!), we can replace it with whatever we want. So, let's call it α, and we get

$$\frac{d\mathbf{x}}{d\tau} = \frac{dx^\alpha}{d\tau}\hat{e}_\alpha + x^\mu \Gamma^\alpha{}_{\mu\nu}\hat{e}_\alpha \frac{dx^\nu}{d\tau}$$

We can factor out the $\hat{e}_\alpha$ term from each to obtain

$$\frac{d\hat{e}_\mu}{d\tau} = \left(\frac{dx^\mu}{d\tau} + x^\mu \Gamma^\alpha{}_{\mu\nu}\frac{dx^\nu}{d\tau}\right)\hat{e}_\alpha$$

This is a measure of the total four-velocity vector. Taking the $\frac{d}{d\tau}$ derivative of the component, we obtain the acceleration,

$$\frac{d^2\mathbf{x}}{d\tau^2} = \mathbf{a}$$

Demanding that all the components of $\mathbf{a}$ are zero, we end up with the expression:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{1}$$

This is called the **geodesic equation**, and it describes straight lines in a space(time) of any curvature!

3.1 Flat Space in Cartesian Coordinates

Let's do a couple simple examples, as always starting with simple two-dimensional flat spaces. In Cartesian coordinates, the metric is $g_{\mu\nu} = \text{diag}(+1, +1)$, and we define the position and acceleration vectors as $\mathbf{x}(t) = (x(t), y(t))$, $\mathbf{a}(t) = (a_x(t), a_y(t))$. Since this isn't spacetime, we can get away with replacing $d\tau \to dt$ without a problem. Also, since the metric components don't depend on any coordinates, the Christoffel symbols must all vanish! So,the geodesic equation in Cartesian coordinates is

$$\frac{d^2x^\alpha}{dt^2} = 0$$

That is, $(a_x(t), a_y(t)) = (0, 0)$. The acceleration in each direction is 0, and so the object is moving in a straight line. The path described by the (non-)acceleration is one whose coordinate dependence has vanishing second derivative, which happens to be

$$x(t) = v_x t \quad , \quad y(t) = v_y t$$

where (v_x, v_y) are constamts. In the absence of forces, objects move in straight lines. How quaintly Newtonian!

3.2 Flat Space in Polar Coordinates

Now let's define our space by the metric $ds^2 = dr^2 + r^2 d\theta^2$, and so $g_{\mu\nu} = \text{diag}(1, r^2)$. The coordinates are now $\mathbf{x}(t) = (r(t), \theta(t))$. As we know, there are two non-vanishing Christoffel symbols:

$$\Gamma^1{}_{22} = -r^2 \quad , \quad \Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r}$$

The two geodesic equations are thus

$$\frac{d^2x^1}{dt^2} + \Gamma^1{}_{22}\frac{dx^2}{dt}\frac{dx^2}{dt} = 0$$

$$\implies \quad \frac{d^2r}{dt^2} - r^2\left(\frac{d\theta}{dt}\right)^2 = 0$$

and

$$\frac{d^2x^2}{dt^2} + \Gamma^2{}_{12}\frac{dx^1}{dt}\frac{dx^2}{dt} + \Gamma^2{}_{21}\frac{dx^2}{dt}\frac{dx^1}{dt} = 0$$

$$\implies \quad \frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} = 0$$

Rearranging them and putting them together, we find

$$\frac{d^2r}{dt^2} = r^2\left(\frac{d\theta}{dt}\right)^2 \quad , \quad \frac{d^2\theta}{dt^2} = -\frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt}$$

to ensure we have no acceleration, we can look at this a few ways. First, the trivial case is that we want $\frac{d^2r}{dt^2}$ and $\frac{d^2\theta}{dt^2}$ to be zero. But this then constrains the values of $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$ as follows:

$$\frac{d^2r}{dt^2} = 0 \quad \Longrightarrow \quad \frac{d\theta}{dt} = 0$$

But if this is true, then the θ-acceleration is

$$\frac{d^2\theta}{dt^2} = 0$$

as well, because it depends on $\frac{d\theta}{dt}$. That's good, because it's consistent!... but what does it tell us about a straight line in polar coordinates??? Note that the above equation **does not constrain** the derivative $\frac{dr}{dt}$. It could be anything, so long as it's constant. And there you have it! An object moving along a radial path $r(t)$ with **constant** velocity $\frac{dr}{dt}$ traces out a straight line in polar coordinates.

4 Curvature as a Measure of Geodesic Deviation

Curvature is also instrumental in describing the behavior of straight lines in spacetime. Recall that a **geodesic** is the straightest possible path one can draw between any two points on a surface or in space(time). Objects that follow geodesics are said to be in **free-fall**, since they are in the absence of external forces or fields. The shape of geodesics can be extracted from the **geodesic equation**

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

which produces a set of parametric, coupled equations for each coordinate $x^\mu(\tau)$.
So, let's consider two objects in free-fall, so that each follows its own geodesic (we'll assume the objects do not interact with each other in any way). Suppose the first object is described by the position coordinates x_1^μ, and the second by $y^\mu = x^\mu + w^\mu$. Here, the vector $\mathbf{w}$ is the separation vector – how it varies as the objects move along their respective geodesics will tell us about the curvature:

$$\frac{d\mathbf{w}}{d\tau} = \left(\frac{dw^\sigma}{d\tau} + \Gamma^\sigma{}_{\alpha\beta}w^\alpha\frac{dx^\beta}{d\tau}\right)\hat{e}_\sigma$$

which we see is explicitly related to the object's four-velocity $u^\mu = \frac{dx^\mu}{d\tau}$ as it moves along its geodesic. Since the geodesic equation is based on the second derivative of the position vectors (*i.e.* acceleration), we can take the second derivative of $\mathbf{n}$ to get

$$\frac{d^2\mathbf{w}}{d\tau^2} = \frac{d}{d\tau}\left(\frac{d\mathbf{w}}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{dw^\sigma}{d\tau} + \Gamma^\sigma{}_{\alpha\beta}w^\alpha\frac{dx^\beta}{d\tau}\right)\hat{e}_\sigma$$

The full derivation is incredibly long and somewhat convoluted, so I'll save you the anguish. The upshot of this is that we derive an equation for the coefficients of the second derivative of the separation vector as a linear combination of its own components n^ν and the components of the object's four-velocity u^α,

$$\left(\frac{d^2\mathbf{w}}{d\tau^2}\right)^\alpha = -R^\alpha{}_{\beta\mu\nu}u^\beta w^\mu u^\nu$$

What are the take-home lessons here? Geodesic deviation tells us:

- The separation between adjacent geodesics changes if there is curvature!
- If the curvature is *positive*, geodesics will **converge**: $\left(\frac{d^2\mathbf{n}}{d\tau^2}\right)^\alpha < 0$
- If the curvature is *negative*, geodesics will **diverge**. $\left(\frac{d^2\mathbf{n}}{d\tau^2}\right)^\alpha > 0$
- If the curvature is *zero* (flat), geodesics will **remain parallel**: $\left(\frac{d^2\mathbf{n}}{d\tau^2}\right)^\alpha = 0$

5 The Painful Derivation of the Equation of Geodesic Deviation

Are you still reading? Good! You must be dying to know *where* the geodesic deviation equation comes from. It'd be my pleasure to show you!

Consider two initially parallel geodesics described by the equations

$$\begin{array}{ll} \boxed{1} & \ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \quad , \\ \boxed{2} & \ddot{y}^\mu + \tilde{\Gamma}^\mu_{\alpha\beta}\dot{y}^\alpha\dot{y}^\beta = 0 \end{array} \tag{2}$$

where the two four-vectors x^μ and y^μ are separated by $w^\mu = y^\mu - x^\mu$, where w^μ is a very small vector. The Christoffel symbol $\tilde{\Gamma}^\mu_{\alpha\beta}$ is evaluated at the coordinate y^μ, and if the two geodesics are sufficiently close, we can approximate this by a Taylor series

$$\tilde{\Gamma}^\mu_{\alpha\beta} \approx \Gamma^\mu_{\alpha\beta} + \partial_\nu\Gamma^\mu_{\alpha\beta}w^\nu$$

If we substitute this and $y^\mu = x^\mu + w^\mu$, then we can rewrite the second geodesic equation for y^μ ($\boxed{2}$) as

$$0 = \ddot{x}^\mu + \ddot{w}^\mu + \left(\Gamma^\mu_{\alpha\beta} + \partial_\nu\Gamma^\mu_{\alpha\beta}w^\nu\right)(\dot{x}^\alpha + \dot{w}^\alpha)(\dot{x}^\beta + \dot{w}^\beta)$$

Since this is a Taylor expansion, we can make some approximations and ignore any terms of order $w^2, w\dot{w}$, *etc.*... Keeping that in mind, the above expands out to

$$0 = \ddot{x}^\mu + \ddot{w}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + 2\Gamma^\mu_{\alpha\beta}\dot{w}^\alpha\dot{x}^\beta + \partial_\nu\Gamma^\mu_{\alpha\beta}w^\nu\dot{x}^\alpha\dot{x}^\beta$$

We can immediately see the geodesic equation $\boxed{1}$ in there, so let's subtract that off! We're left with

$$0 = \ddot{w}^\mu + 2\Gamma^\mu_{\alpha\beta}\dot{w}^\alpha\dot{x}^\beta + \partial_\nu\Gamma^\mu_{\alpha\beta}w^\nu\dot{x}^\alpha\dot{x}^\beta$$

This gives us an expression for $\ddot{w}^\mu$,

$$\ddot{w}^\mu = -2\Gamma^\mu_{\alpha\beta}\dot{w}^\alpha\dot{x}^\beta - \partial_\nu\Gamma^\mu_{\alpha\beta}w^\nu\dot{x}^\alpha\dot{x}^\beta \tag{3}$$

which will come in handy later.
So moving on, let's now consider how the vector $\mathbf{w} = w^\mu\hat{e}_\mu$ changes with proper time:

$$\frac{d\mathbf{w}}{d\tau} = \frac{dw^\mu}{d\tau}\hat{e}_\mu + w^\mu\frac{d\hat{e}_\mu}{d\tau}$$

The second derivative can be expanded out as

$$\frac{d\hat{e}_\mu}{d\tau} = \frac{\partial\hat{e}_\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}$$

and that looks familiar! Remember when we first discussed Christoffel symbols, it came from the definition $\partial_\nu\hat{e}_\mu = \Gamma^\alpha_{\mu\nu}\hat{e}_\alpha$. So, the above derivative for $\mathbf{w}$ is actually

$$\begin{aligned}\frac{d\mathbf{w}}{d\tau} &= \frac{dw^\mu}{d\tau}\hat{e}_\mu + \Gamma^\alpha_{\mu\nu}w^\mu\dot{x}^\nu\hat{e}_\alpha \\ &= \left[\dot{w}^\alpha + \Gamma^\alpha_{\mu\nu}w^\mu\dot{x}^\nu\right]\hat{e}_\alpha \\ &= \left[\dot{w}^\mu + \Gamma^\mu_{\alpha\beta}w^\alpha\dot{x}^\beta\right]\hat{e}_\mu\end{aligned}$$

where a bunc of index swaps were made so this is consistent with the earlier equations (remember: you can always call dummy indicies whatever you want!). If we take the proper time derivative of this again, we get

$$\begin{aligned}\frac{d^2\mathbf{w}}{d\tau^2} &= \frac{d}{d\tau}\left(\left[\dot{w}^\mu + \Gamma^\mu_{\alpha\beta}w^\alpha\dot{x}^\beta\right]\hat{e}_\mu\right) \\ &= \left[\ddot{w}^\mu + \Gamma^\mu_{\alpha\beta}\dot{w}^\alpha\dot{x}^\beta + \Gamma^\mu_{\alpha\beta}w^\alpha\ddot{x}^\beta + \partial_\sigma\Gamma^\mu_{\alpha\beta}w^\alpha\dot{x}^\beta\dot{x}^\sigma\right]\hat{e}_\mu \\ &\quad + \left[\dot{w}^\mu + \Gamma^\mu_{\alpha\beta}w^\alpha\dot{x}^\beta\right]\Gamma^\sigma_{\mu\nu}\dot{x}^\nu\hat{e}_\sigma\end{aligned} \tag{4}$$

This contains a $\ddot{w}^\mu$, so we can use Equation (3) in Equation (4)! If we do that, we find

$$\begin{aligned}\frac{d^2\mathbf{w}}{d\tau^2} &= \left[-2\Gamma^\mu_{\alpha\beta}\dot{w}^\alpha\dot{x}^\beta - \partial_\sigma\Gamma^\mu_{\alpha\beta}w^\sigma\dot{x}^\alpha\dot{x}^\beta + \Gamma^\mu_{\alpha\beta}\dot{w}^\alpha\dot{x}^\beta + \Gamma^\mu_{\alpha\beta}w^\alpha\ddot{x}^\beta + \partial_\sigma\Gamma^\mu_{\alpha\beta}w^\alpha\dot{x}^\beta\dot{x}^\sigma\right]\hat{e}_\mu \\ &\quad + \left[\dot{w}^\mu + \Gamma^\mu_{\alpha\beta}w^\alpha\dot{x}^\beta\right]\Gamma^\sigma_{\mu\nu}\dot{x}^\nu\hat{e}_\sigma\end{aligned} \tag{5}$$

Since $-2\Gamma\dot{w}\dot{x} + \Gamma\dot{w}\dot{x} = -\Gamma\dot{w}\dot{x}$, we can mildly simplify this to

$$\begin{aligned}\frac{d^2\mathbf{w}}{d\tau^2} &= \left[-\Gamma^\mu_{\alpha\beta}\dot{w}^\alpha\dot{x}^\beta - \partial_\sigma\Gamma^\mu_{\alpha\beta}w^\sigma\dot{x}^\alpha\dot{x}^\beta + \Gamma^\mu_{\alpha\beta}w^\alpha\ddot{x}^\beta + \partial_\sigma\Gamma^\mu_{\alpha\beta}w^\alpha\dot{x}^\beta\dot{x}^\sigma\right]\hat{e}_\mu \\ &\quad + \left[\dot{w}^\mu + \Gamma^\mu_{\alpha\beta}w^\alpha\dot{x}^\beta\right]\Gamma^\sigma_{\mu\nu}\dot{x}^\nu\hat{e}_\sigma\end{aligned} \tag{6}$$

But it's still quite a mess! Yet we persevere and make it messier. The above expression contains an $\ddot{x}$ term, so we can substitute in the original geodesic equation (2), solving for $\ddot{x}$, to get the term

$$\Gamma^{\mu}_{\alpha\beta} w^{\alpha} \ddot{x}^{\beta} = -\Gamma^{\mu}_{\alpha\beta} \Gamma^{\beta}_{\sigma\tau} w^{\alpha} \dot{x}^{\sigma} \dot{x}^{\tau}$$

And so,

$$\begin{aligned} \frac{d^2\mathbf{w}}{d\tau^2} &= \left[-\Gamma^{\mu}_{\alpha\beta} \dot{w}^{\alpha} \dot{x}^{\beta} - \partial_{\sigma} \Gamma^{\mu}_{\alpha\beta} w^{\sigma} \dot{x}^{\alpha} \dot{x}^{\beta} + -\Gamma^{\mu}_{\alpha\beta} \Gamma^{\beta}_{\sigma\tau} w^{\alpha} \dot{x}^{\sigma} \dot{x}^{\tau} + \partial_{\sigma} \Gamma^{\mu}_{\alpha\beta} w^{\alpha} \dot{x}^{\beta} \dot{x}^{\sigma} \right] \hat{e}_{\mu} \\ &+ \left[\Gamma^{\sigma}_{\mu\nu} \dot{w}^{\mu} \dot{x}^{\nu} + \Gamma^{\mu}_{\alpha\beta} \Gamma^{\sigma}_{\mu\nu} w^{\alpha} \dot{x}^{\beta} \dot{x}^{\nu} \right] \hat{e}_{\sigma} \qquad (7) \end{aligned}$$

Aha! Note the first term

$$-\Gamma^{\mu}_{\alpha\beta} \dot{w}^{\alpha} \dot{x}^{\beta} \hat{e}_{\mu}$$

is the opposite of the last second to last term

$$\Gamma^{\sigma}_{\mu\nu} \dot{w}^{\mu} \dot{x}^{\nu} \hat{e}_{\sigma}$$

So they cancel! That means we're left with

$$\begin{aligned} \frac{d^2\mathbf{w}}{d\tau^2} &= \left[-\partial_{\sigma} \Gamma^{\mu}_{\alpha\beta} w^{\sigma} \dot{x}^{\alpha} \dot{x}^{\beta} - \Gamma^{\mu}_{\alpha\beta} \Gamma^{\beta}_{\sigma\tau} w^{\alpha} \dot{x}^{\sigma} \dot{x}^{\tau} + \partial_{\sigma} \Gamma^{\mu}_{\alpha\beta} w^{\alpha} \dot{x}^{\beta} \dot{x}^{\sigma} \right] \hat{e}_{\mu} \\ &+ \Gamma^{\mu}_{\alpha\beta} \Gamma^{\sigma}_{\mu\nu} w^{\alpha} \dot{x}^{\beta} \dot{x}^{\nu} \hat{e}_{\sigma} \qquad (8) \end{aligned}$$

WAIT A SEC! This looks **very** familiar! Two terms with $\partial\Gamma$, one positive and the other negative, and two terms with $\Gamma\Gamma$ – one positive and the other negative! I think it's the **Riemann curvature tensor!**

In fact, it **is** the Riemann curvature tensor! If we take the last term and replace $\sigma \to \mu, \mu \to \beta, \nu \to \alpha, \alpha \to \sigma$, and $\beta \to \tau$, we'll get

$$\Gamma^{\mu}_{\alpha\beta} \Gamma^{\beta}_{\sigma\tau} w^{\sigma} \dot{x}^{\tau} \dot{x}^{\alpha} \hat{e}_{\mu}$$

Comparing this to the other $\Gamma\Gamma$ term, we see

$$\Gamma^{\mu}_{\alpha\beta} \Gamma^{\beta}_{\sigma\tau} w^{\sigma} \dot{x}^{\tau} \dot{x}^{\alpha} \hat{e}_{\mu} - \Gamma^{\mu}_{\alpha\beta} \Gamma^{\beta}_{\sigma\tau} w^{\alpha} \dot{x}^{\sigma} \dot{x}^{\tau} \hat{e}_{\mu}$$

Lastly, if we demand that the vector terms have the same index pattern $w^{\alpha} \dot{x}^{\sigma} \dot{x}^{\tau}$, then this becomes

$$\left(\Gamma^{\mu}_{\tau\beta} \Gamma^{\beta}_{\alpha\sigma} - \Gamma^{\mu}_{\alpha\beta} \Gamma^{\beta}_{\sigma\tau} \right) w^{\alpha} \dot{x}^{\sigma} \dot{x}^{\tau} \hat{e}_{\mu}$$

Doing the same thing for the $\partial\Gamma$ terms, we have

$$-\partial_{\sigma} \Gamma^{\mu}_{\alpha\beta} w^{\sigma} \dot{x}^{\alpha} \dot{x}^{\beta} \implies -\partial_{\alpha} \Gamma^{\mu}_{\sigma\tau} w^{\alpha} \dot{x}^{\sigma} \dot{x}^{\tau}$$

and

$$\partial_{\sigma} \Gamma^{\mu}_{\alpha\beta} w^{\alpha} \dot{x}^{\beta} \dot{x}^{\sigma} \implies \partial_{\tau} \Gamma^{\mu}_{\alpha\sigma} w^{\alpha} \dot{x}^{\sigma} \dot{x}^{\tau}$$

Putting them all together in the original expression, we find

$$\frac{d^2\mathbf{w}}{d\tau^2} = \left[-\partial_\alpha \Gamma^\mu_{\sigma\tau} + \partial_\tau \Gamma^\mu_{\alpha\sigma} + \Gamma^\mu_{\tau\beta}\Gamma^\beta_{\alpha\sigma} - \Gamma^\mu_{\alpha\beta}\Gamma^\beta_{\sigma\tau}\right] w^\alpha \dot{x}^\sigma \dot{x}^\tau \hat{e}_\mu$$

IT IS THE CURVATURE TENSOR! It's customary to write this as

$$\frac{d^2 w^\alpha}{d\tau^2} = -\left[\partial_\alpha \Gamma^\mu_{\sigma\tau} - \partial_\tau \Gamma^\mu_{\alpha\sigma} + \Gamma^\mu_{\alpha\beta}\Gamma^\beta_{\sigma\tau} - \Gamma^\mu_{\tau\beta}\Gamma^\beta_{\alpha\sigma}\right] w^\alpha \dot{x}^\sigma \dot{x}^\tau \hat{e}_\mu$$

which is the curvature tensor $R^\mu{}_{\sigma\alpha\tau}$. And so, the equation for geodesic deviation is

$$\frac{d^2\mathbf{w}}{d\tau^2} = -R^\mu{}_{\sigma\alpha\tau} w^\alpha \dot{x}^\sigma \dot{x}^\tau$$

PHEW!