Einstein's Equations and the Schwarzschild Metric

PHYS 471: Introduction to Relativity and Cosmology

1 Curvature Recap

Over the past few weeks, you have become extremely familiar with the concept of curvature, as well as its mathematical description. *Why* you have done so, however, is still a mystery! If you've seen the classic 80s movie *The Karate Kid*, you'll remember that Mr. Miyagi made Daniel wax his car instead of teaching him karate. But in an impromptu sparring session, the orders "WAX ON!" and "WAX OFF!" revealed that Daniel had become an EXPERT in karate without realizing it! This is your WAX ON!, WAX OFF! moment in relativity....

Our study of curvature so far has involved the following important objects:

$$
\begin{aligned}
\text{Metric tensor} \quad &\Longrightarrow \quad g_{\mu\nu} \\
\text{Riemann curvature tensor} \quad &\Longrightarrow \quad R^{\alpha}{}_{\beta\mu\nu} = g^{\alpha\sigma} R_{\sigma\beta\mu\nu} \\
\text{Ricci tensor} \quad &\Longrightarrow \quad R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} \\
\text{Ricci scalar} \quad &\Longrightarrow \quad R = g^{\mu\nu} R_{\mu\nu}
\end{aligned}
$$

We have also seen that objects moving along geodesics in curved spaces end up looking like they're accelerating, which isn't what happens in flat spacetime. Einstein rationalized that the secret to gravity was curvature, and that objects under the exclusive influence of gravity– *i.e.* in free fall– followed geodesics in a curved spacetime.

2 The Equivalence Principle

Before getting too far along with Einstein, let's stop to reminisce about Newtonian gravity, and in doing so we'll eventually relate it to Einstein's new perspective. You of course know that the net force acting on an object is $\vec{F} = m\vec{a}$, and the gravitational force acting on an object is $\vec{F}_g = m\vec{g}$, so of course it follows that

$$\vec{F} = m\vec{a} \quad , \quad \vec{F}_g = m\vec{g}$$

$$\Longrightarrow \quad \vec{F} = \vec{F}_g \quad \Longrightarrow \quad \vec{a} = \vec{g}$$

Right? WRONG!!

The problem here is that we can't immediately claim the masses in those equations are the same! One is the mass that responds to an arbitrary force, the other is the mass that responds to a gravitational force. We'll call these the **inertial mass** m

and the **gravitational mass** m_g, respectively. If we insert those definitions into the above equation, we find that

$$m\vec{a} = m_g\vec{g}$$

which does *not* say that $\vec{g}$ is the acceleration, any more that the acceleration of a charged particle under the influence of an electric field is the same as the electric field:

$$\vec{F} = \vec{F}_E \quad \Longrightarrow \quad m\vec{a} = q\vec{E}$$

"But wait!", you say, "Of course $\vec{E}$ is not the acceleration – it's the electric field!" And so, I say to you: "OK! In that case, let's call $\vec{g}$ is the **gravitational field**, and m_g is the 'gravitational charge' that responds to it." This sounds good to me! So, from now on, you will always call $\vec{g}$ the "gravitational field strength", and not the acceleration due to gravity!

The oft-made confusion between $\vec{g}$ as an acceleration and $\vec{g}$ as a field comes from the fact that, to as high a precision as we've tested it, there appears to be virtually **no difference between inertial mass and gravitational mass!** When I did my thesis ... *a few* years ago... the current state of the art was that the two were the same up to $\frac{|m-m_g|}{m} \approx 10^{-12}$ or so! That these two unrelated masses are essentially the same is called **the Equivalence Principle**, and it was this conundrum that led Einstein to start thinking about gravitation in a new way. Why? Because he realized that iif $m = m_g$, an observer in **free fall** in a gravitational field would feel no acceleration,

$$m\vec{a} - m_g\vec{g} = 0 \quad \Longrightarrow \vec{a} = \vec{g}$$

That is, unlike the case of E-fields, gravitation had the special property that anyone of any mass could "turn it off" if we followed special paths in spacetime...

3 Gravitation as a Classical Field

Since we can describe gravitational as a force resulting from a field, we can make some comparisons to the E-field in E&M. As you know, Coulomb's law bears a striking resemblance to Newtons' law of gravitation,

$$\vec{F}_E = \frac{kQq}{r^2}\hat{r} \quad , \quad \vec{F}_g = -\frac{GMm}{r^2}\hat{r}$$

and so the respective fields are

$$\vec{E} = \frac{kQ}{r^2}\hat{r} \quad , \quad \vec{g} = -\frac{GM}{r^2}\hat{r}$$

In E&M, we know that Gauss' Law tells us something about the behavior of radial fields, namely

$$\vec{\nabla} \cdot \vec{E} = 4\pi k\rho_E \quad \Longrightarrow \quad \mathrm{E-fields\ diverge\ from\ charge\ distributions}$$

where ρ_E is the charge density, and the charge is

$$Q = \int \rho_E \, dV$$

Another way of interpreting Gauss' Law is to say "Electric charges create E-fields."

By complete analogy, we can say that gravitational fields diverge from (and are created by) mass distributions,

$$\vec{\nabla} \cdot \vec{g} = -4\pi G \rho_M \quad \Longrightarrow \quad \text{g} - \text{fields diverge from mass distributions}$$

Here, ρ_M is the mass density, and the mass is

$$M = \int \rho_M \, dV$$

Furthermore, just as the E-field is determined by the gradient of the electric potential, the gravitational field is determined as the gradient of the gravitational potential:

$$\vec{E} = -\vec{\nabla}\phi_E(r) \;\; , \;\; \phi_E = \frac{kQ}{r} \quad \Longrightarrow \quad \vec{g} = -\vec{\nabla}\phi_g(r) \;\; , \;\; \phi_g = -\frac{GM}{r}$$

In E&M, the divergence of the gradient of the potential is given by the Poisson equation,

$$\vec{\nabla} \cdot \vec{E} = 4\pi k \rho_E \quad \Longrightarrow \quad \nabla^2 \phi_E = 4\pi k \rho_E$$

and it tells us that charges generate electric potentials, which in turn generate E-fields. As before, we can re-write Gauss' Law for gravitational fields as

$$\vec{\nabla} \cdot \vec{g} = 4\pi G \rho_M \quad \Longrightarrow \quad \nabla^2 \phi_G = 4\pi G \rho_M$$

This equation is very important: it says that **the gravitational field is generated by mass**, which you probably knew. More specifically, it says that **the second derivative of the "source" of the gravitational field is related to the mass that generates it**:

$$\text{Source of gravitational field} \;=\; \text{Mass}$$

4 Einstein Sez: "Gravity is Curvature"!

Let?s now turn this into Einstein thinking! The source of the gravitational field is no longer a scalar potential function, but rather curvature!. So, the idea expressed above can be carried over to relativity as follows:

$$\text{Source of gravitational field} \;=\; \text{Curvature}$$

Combining this with the Newtonian version, we get the paradigm shifting conclusion Einstein reached:

$$\text{Curvature} = \text{Mass} \tag{1}$$

This is the fundamental tenet of Einstein's equations: **curvature of spacetime is generated by mass.**

We're now in a position to start building Einstein?s equations! Taking inspiration from Newtonian theory, it would be great if we could find an equation that depends on **the second derivatives of the source of curvature**, just as Newtonian gravity depends on the second derivative of the source of the field. Of course, we already have something that fits this bill: **the curvature tensor!**

We need to equate this to something that tells us about mass (density) distributions in spacetime. Just as there is a stress-energy tensor for E&M ($F^{\mu\nu}$), there is one for mass/energy as well:

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + p g^{\mu\nu}$$

This is a 4×4 diagonal matrix whose entries are *energy density in time* (ρ), and *the energy density in space*, which is nothing more than pressure $\vec{p}$! (It's true – check the units of pressure. It's a measure of energy density). The velocities u^μ and u^ν describe relative motion of an observer in those particular direction. An object's mass is therefore the spatial integral of this component,

$$M = \int T_{00}\, dV$$

which is the relativistic equivalent of the Newtonian definition.

5 Building Einstein's Equations

If matter/energy is described by a matrix, then we want curvature to be so as well. Of course, we have such a beast, **the Ricci tensor**. So, our first attempt at writing an equation to describe gravity is

$$\text{Attempt 1}: \quad \longrightarrow \quad R_{\mu\nu} \propto T_{\mu\nu}$$

We haven't set them equal yet, because we need to adjust the units. But fundamentally, these two objects are proportional.

It turns out that we can add *more* terms to the LHS. In addition to the Ricci tensor, we can also describe curvature in terms of this 44 matrix: $Rg_{\mu\nu}$. That is, the Ricci scalar times the metric. So, our equation could read

$$\text{Attempt 2 :} \quad \longrightarrow \quad R_{\mu\nu} + g_{\mu\nu}R \propto T_{\mu\nu}$$

And last, but not least, there is a *third* term we can add to the LHS to make the equation as general as possible. In addition to the Ricci scalar, we can add a generic constant multiple of the metric: $\Lambda g_{\mu\nu}$. This makes our equation look like

$$\text{Attempt 3 :} \quad \longrightarrow \quad R_{\mu\nu} + g_{\mu\nu}R + \Lambda g_{\mu\nu} \propto T_{\mu\nu}$$

There it is! Adding in all the necessary constants, and throwing in a few minus signs for good measure, we get

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{2}$$

These are **the Einstein Field Equations**. The above equations represent a 4×4 matrix of second-order partial differential equations in the metric $g_{\mu\nu}$, and since we already know $R_{\mu\nu}$ has 10 independent quantities, there must be 10 equations here! The curvature terms contain the metric derivatives, while the last term Λ is called the **cosmological constant**.

Looking back to the Newtonian case, we see this makes sense. In the Newtonian case, we know the gravitational field is the gradient of the potential,

$$\vec{g} = -\vec{\nabla}\phi_G$$

and the Poisson equation tells us that the second derivative of the potential is proportional to the source,

$$\nabla^2\phi_G = 4\pi G\rho_M$$

We can sum this up with the statement

$$\text{Newtonian gravity} \quad \Longrightarrow \quad \nabla^2 \text{ of gravitational potential } \propto \text{ matter source}$$

Now from the Einstein perspective, the field equations are in terms of curvature, which comes from second-order covariant derivatives of the metric! And since the source of curvature is matter, we can say

$$\text{Einstein gravity} \quad \Longrightarrow \quad D^2 \text{ of metric } \propto \text{ matter source}$$

COOOL!!!

6 The First Solution: The Schwarzschild Metric

Einstein's equations are second-order partial differential equations for the metric, whose boundary conditions are fixed by the source – that is, the distribution of matter! So, when we solve Einstein's equations for a specific density distribution, we get the metric. In the absence of any matter, the solution is the flat spacetime metric, $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ because there is no curvature. If we want to know how spacetime acts around some mass M, however, we get something decidedly different. Clearly, the spacetime must curve, since any mass creates a gravitational field.

The first solution to Einstein?s equations came only a few years after Einstein actually proposed them, and as luck would have it it *wasn't* by Einstein himself! Instead, a German physicist by the name of **Karl Schwarzschild** derived it, beating out Einstein because he closed the elevator door on him before he could get in. Actually, Schwarzschild was serving in the German army at the time of his derivation, stationed on the front lines of World War I! He asked himself what the most general, **spherically-symmetric metric would be for a static mass**. Using the stress-energy tensor for this case, with a mass M located at the origin of a spherical spacetime coordinate system, he found the following metric:

$$g_{\mu_n u} = \begin{pmatrix} 1 - \frac{2GM}{rc^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{2GM}{rc^2}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

which is alternatively expressed as the line element

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) dt^2 - \frac{dr^2}{1 - \frac{2GM}{rc^2}} - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \qquad (3)$$

7 Consequences of the Schwarzschild Solution

We can infer some interesting properties of the spacetime described by the Schwarzschild solution. First and foremost, it is **spherically symmetric**. This is clear because there is no coordinate dependence in the θ and $d\phi$ components, g_{22} and g_{33}. In fact, it looks exactly like the metric for a spherical space.

The difference arises in the t and r components, through the function

$$g_{00} = -g_{11}^{-1} = 1 - \frac{2GM}{rc^2}$$

These change as one moves radially from the mass. If we're very far away from it, the function in the metric approaches unity

$$\lim_{r\to\infty}\left(1-\frac{2GM}{rc^2}\right)=1$$

and so the metric reduces to

$$ds^2 = dt^2 - dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2$$

It's the flat spacetime metric! We say the Schwarzschild solution is **asymptotically flat**, which is just a fancy way of saying that the effects of gravity vanish infinitely far away from the source. But of course, we already knew this from Newton!

But the more interesting behavior comes in the limit of small r. If we dial it down so $r \to 0$, the function in the components diverges!

$$\lim_{r\to 0}\left(1-\frac{2GM}{rc^2}\right)\to\infty$$

We call this a **singularity**, and it tells us that something is very wrong with this solution at $r = 0$. Things in physics don't do infinity! This is a huge problem in general relativity, and we'll discuss it at much greater length shortly.

An even more interesting thing happens for the radius at which the function vanishes:

$$1-\frac{2GM}{rc^2}=0 \quad\Longrightarrow\quad r=R_S=\frac{2GM}{c^2}$$

At this point, the time component g_{00} of the metric vanishes, and the radial component g_{11} diverges! This is called the **Schwarzschild radius**, and some very, very, *VERY* strange physics happens here.

Typical objects – planets, stars, galaxies, you – have Schwarzschild radii that are much smaller than their actual size. The Schwarzschild radius of the Sun, for example, is

$$\text{Sun} \quad\longrightarrow\quad R_S=\frac{2(6.67\times10^{-11})(2\times10^{30})}{(3\times10^{8})^2}\approx 3000\text{ m}$$

The Sun is clearly larger than 3 km! Likewise, the Schwarzschild radius of the Earth is

$$\text{Earth} \quad\longrightarrow\quad R_S=\frac{2(6.67\times10^{-11})(6\times10^{24})}{(3\times10^{8})^2}\approx 0.009\text{ m}$$

The Earth is clearly larger than one millimetre! Just for fun, your Schwarzschild radius is about

$$\text{You} \quad\longrightarrow\quad R_S\approx 10^{-25}\text{ m}$$

So, what's the point of calculating this radius? And what's so strange about it? Everything in the Universe is much bigger than it.... Or are they???!!! What if there are objects so dense that *they sit entirely within their Schwarzschild radius*? They exist!.... They are out there!.... and they are coming for us!!

They are... BLACK HOLES!