Christoffel Symbols
PHYS 471: Introduction to Relativity and Cosmology

1 The Metric and Coordinate Basis

Let's recap some properties of the metric, before deriving the Christoffel symbols. First, we know the metric is a tensor $g_{\mu\nu}$, which can be written as a 4×4 diagonal matrix. The line element ds^2 is the length-squared of an infinitesimal vector in spacetime:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

The components of the metric will depend on the coordinate system, however. For flat spacetime in Cartesian coordinates, this is the familiar Minkowski metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad , \quad g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$$

But if we were to change coordinates to, say, spherical ones, the line element would look different:

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

These metric coefficients can be "encoded" in what's called a "basis vector" $\hat{e}_\mu$ (more technically a *vierbein*!). Formally, we define the metric as the product

$$g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu \tag{2}$$

Great, but what *is* this basis vector thing?? You've actually seen them before. When you first start talking about vectors, we define the unit vectors $\hat{x}, \hat{y}, \hat{z}$ that point in those orthogonal directions. A general vector is then written

$$\vec{s} = A^x \hat{x} + A^y \hat{y} + A^z \hat{z}$$

When we change coordinates to spherical ones, we want to re-express these coefficients as some other set

$$\vec{s} = A^r \hat{r} + A^\theta \hat{\theta} + A^\phi \hat{\phi}$$

So, consider a vector that has infinitesimal length for all its components: $A^x = dx, A^y = dy, A^z = dz$. Then,

$$\vec{ds} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

If we switch to spherical coordinates, the appropriate transformation (using the Jacobian) gives

$$\vec{ds} = dr\,\hat{r} + r d\theta\,\hat{\theta} + r\sin\theta\hat{} d\phi\phi$$

But what we really want to do is express *any* such vector in a way that "looks like" the Cartesian version:

$$\vec{ds} = d\hat{x}_1 + d\hat{x}_2 + d\hat{x}_3$$

In order to do so, we define the **coordinate basis vector component** $\hat{e}_i$ to be the thing that allows us to write

$$\vec{ds} = d\hat{x}_1 + d\hat{x}_2 + d\hat{x}_3 = dx_1\hat{e}_1 + dx_2\hat{e}_2 + dx_3\hat{e}_3$$

Note that the "basis" quality has been taken away from dx_i, which is now only a number. So, for the Cartesian system, $d\hat{x}_1 = dx\,\hat{e}_x$, $d\hat{x}_2 = dy\,\hat{e}_y$, $d\hat{x}_3 = dz\,\hat{e}_z$, which tells us that the components of the coordinate basis vector are $\hat{e}_x = \hat{e}_y = \hat{e}_z = 1$.

Using this formalism, the line element is

$$ds^2 = \vec{ds}\cdot\vec{ds} = (\hat{e}_1)^2 dx_1^2 + (\hat{e}_2)^2 dx_2^2 + (\hat{e}_3)^2 dx_3^2$$

Comparing the above to Equations **??** and **??**, we see that $g_{11} = \hat{e}_1 \cdot \hat{e}_1 = (\hat{e}_1)^2$, and so forth. If we did the same thing in spherical coordinates, we'd find

$$\vec{ds} = d\hat{r} + d\hat{\theta} + d\hat{\phi} = dr\,\hat{e}_r + d\theta\,\hat{e}_\theta + d\phi\,\hat{e}_\phi$$

$$\implies \quad \hat{e}_r = 1 \quad , \quad \hat{e}_\theta = r \quad , \quad \hat{e}_\phi = r\sin\theta$$

$$\implies \quad g_{rr} = (\hat{e}_r)^2 = 1 \quad , \quad g_{\theta\theta} = (\hat{e}_\theta)^2 = r^2 \quad , \quad g_{\phi\phi} = (\hat{e}_\phi)^2 = r^2\sin^2\theta$$

Note that we're used to seeing these as negative, which they still are in spacetime! The above discussion was only for spatial components.

2 Properties of the Christoffel Symbols

The coordinate basis vectors have an obvious coordinate dependence in some systems, so we must account for this when we compute things like derivatives. The **Christoffel symbols** encapsulate this dependence when you take the derivative of a coordinate basis vector. We define this as

$$\partial_\mu \hat{e}_\nu = \Gamma^\lambda{}_{\mu\nu}\hat{e}_\lambda = \Gamma^0{}_{\mu\nu}\hat{e}_0 + \Gamma^1{}_{\mu\nu}\hat{e}_1 + \Gamma^2{}_{\mu\nu}\hat{e}_2 + \Gamma^3{}_{\mu\nu}\hat{e}_3$$

That is, the derivative of the basis vector is a **linear combination** of the four components of the basis vector! This accounts for any motion the basis vector might undergo as it moves around a non-trivial geometry.

Since we know $g_{\mu\nu} = \hat{e}_\mu\hat{e}_\nu$, it's straightforward to show that the derivative of the metric is

$$\partial_\alpha g_{\mu\nu} = (\partial_\alpha \hat{e}_\mu)\hat{e}_\nu + \hat{e}_\mu(\partial_\alpha \hat{e}_\nu) = \Gamma^\lambda{}_{\alpha\mu}\hat{e}_\lambda\hat{e}_\nu + \Gamma^\lambda{}_{\alpha\nu}\hat{e}_\lambda\hat{e}_\mu$$

$$\implies \partial_\alpha g_{\mu\nu} = \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} + \Gamma^\lambda{}_{\alpha\nu} g_{\lambda\mu} \tag{3}$$

From this, we can derive an expression for the Chirstoffel symbols in terms of **first derivatives of the metric**. When we cycle the order of the indices α, μ, ν, we get two more equations:

$$\implies \partial_\mu g_{\nu\alpha} = \Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha} + \Gamma^\lambda{}_{\mu\alpha} g_{\lambda\nu} \tag{4}$$

$$\implies \partial_\nu g_{\alpha\mu} = \Gamma^\lambda{}_{\nu\alpha} g_{\lambda\mu} + \Gamma^\lambda{}_{\nu\mu} g_{\lambda\alpha} \tag{5}$$

As luck would have it (or for geometric reasons, if you prefer), the Christoffel symbols are **symmetric in their lower indices**, which reveals that many of these terms are the same. For example, $\Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha}$ in Equation **??** is the same as $\Gamma^\lambda{}_{\nu\mu} g_{\lambda\alpha}$ in Equation **??**, because we can write $\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}$.

Re-writing the above equations to make this obvious, we get

$$\begin{aligned}
(3) \text{ becomes} \quad &\longrightarrow \quad \partial_\nu g_{\alpha\mu} &=& \quad \Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha} + \Gamma^\lambda{}_{\alpha\nu} g_{\lambda\mu} & (6)\\
(1) \text{ becomes} \quad &\longrightarrow \quad \partial_\alpha g_{\mu\nu} &=& \quad \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} + \Gamma^\lambda{}_{\alpha\nu} g_{\lambda\mu} & (7)\\
(2) \text{ becomes} \quad &\longrightarrow \quad \partial_\mu g_{\nu\alpha} &=& \quad \Gamma^\lambda{}_{\alpha\mu} g_{\lambda\nu} + \Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha} & (8)
\end{aligned}$$

Note each term in Equation **??** (formerly Eq 1) appears in Equation **??** and **??**, respectively. So, we combine these three equations and solve for the Christoffel symbols:

$$\begin{aligned}
& \text{Equation 4} \quad + \quad \text{Equation 6} \quad - \quad \text{Equation 4} \\
= \; & \partial_\nu g_{\alpha\mu} + \partial_\mu g_{\nu\alpha} - \partial_\alpha g_{\mu\nu} \\
= \; & \Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha} + \Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha} \\
= \; & 2\Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha} & (9)
\end{aligned}$$

Success! All we need to do is isolate the Γ, and we do so by multiplying both sides by the inverse metric $g^{\lambda\alpha}$:

$$2\Gamma^\lambda{}_{\mu\nu} g_{\lambda\alpha} = \partial_\nu g_{\alpha\mu} + \partial_\mu g_{\nu\alpha} - \partial_\alpha g_{\mu\nu}$$

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \left(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\lambda\nu} - \partial_\lambda g_{\mu\nu} \right) \tag{10}$$

OK, so now that we have a way to calculate these coefficients, what good are they? As mentioned before, these tell us something about how basis vectors change their direction as a vector moves around some space(time) with curvature. They also tell us something about the nature of lines in the curved spacetime, specifically straight ones (which are kinda sorta important to General Relativity).

3 Uses of the Christoffel Symbols

There are varied applications of the Christoffel symbols in General Relativity, and more broadly in differential geometry. We'll encounter a few key ones in our discussions, however, so let's review three fundamental ones right now.

3.1 The Covariant Derivative

Christoffel symbols are defined in terms of the derivative of the coordinate basis vector components. Their first and most important application is in the computation of derivatives in non-flat spacetimes. If we take the derivative of a vector $\mathbf{A} = A^{\mu}\hat{e}_{\mu}$, we get

$$\partial_{\mu}\mathbf{A} = \partial_{\mu}(A^{\nu}\hat{e}_{\nu}) = (\partial_{\mu}A^{\nu})\hat{e}_{\nu} + A^{\nu}(\partial_{\mu}\hat{e}_{\nu})$$

The second term in the derivative is the Christoffel symbol, so

$$\partial_{\mu}(A^{\nu}\hat{e}_{\nu}) = \partial_{\mu}A^{\nu}\hat{e}_{\nu} + A^{\nu}\Gamma^{\alpha}{}_{\mu\nu}\hat{e}_{\alpha}$$

Since the ν index in the first term is paired, we can arbitrarily replace it with α to find

$$\partial_{\mu}A^{\alpha}\hat{e}_{\alpha} + A^{\nu}\Gamma^{\alpha}{}_{\mu\nu}\hat{e}_{\alpha} = \left(\partial_{\mu}A^{\alpha} + A^{\nu}\Gamma^{\alpha}{}_{\mu\nu}\right)\hat{e}_{\alpha}$$

The term in parenthesis shows how the derivative of the vector $\mathbf{A}$ also depends on the coordinate basis terms. We call this the **covariant derivative of** A^{ν}:

$$\implies \boxed{D_{\mu}A^{\nu} = \partial_{\mu}A^{\nu} + \Gamma^{\nu}{}_{\mu\alpha}A^{\alpha}}$$

Note the lower in the Christoffel symbol combine the derivative index with the vector index (hence, "mixing" the coordinates due to derivatives).
In General Relativity, we also deal with objects with lower indices, though, so some extra care must be taken. It turns out that the derivative of a **covariant vector** A_{μ} looks almost the same, except the sign in front of the Christoffel symbol is *negative*:

$$\implies \boxed{D_{\mu}A_{\nu} = \partial_{\mu}A_{\nu} - \Gamma^{\alpha}{}_{\mu\nu}A_{\alpha}}$$

So the rule is: **upstairs indices get a positive Christoffel symbol, downstairs indices get a negative one**. You can remember this easily, since **up is positive, down is negative**.

Based on these rules, we can take the covariant derivative of a mixed tensor of any type, as long as we have a Christoffel symbol for each index. For example, the covariant derivative of $R^{\mu\nu}$, $R_{\mu\nu}$, and $R^{\mu}{}_{\nu}$ are

$$\text{Positive for upper indices} \quad \implies \quad D_{\alpha}R^{\mu\nu} = \partial_{\alpha}R^{\mu\nu} + \Gamma^{\alpha}{}_{\mu\beta}R^{\beta\nu} + \Gamma^{\alpha}{}_{\beta\nu}R^{\mu\beta}$$

$$\text{Negative for lower indices} \implies D_\alpha R_{\mu\nu} = \partial_\alpha R_{\mu\nu} - \Gamma^\beta{}_{\alpha\mu} R_{\beta\nu} - \Gamma^\beta{}_{\alpha\nu} R_{\mu\beta}$$

$$+ \text{ and } - \text{ for mixed indices} \implies D_\alpha R^\mu{}_\nu = \partial_\alpha R^\mu{}_\nu + \Gamma^\mu{}_{\alpha\beta} R^\beta{}_\nu - \Gamma^\beta{}_{\alpha\nu} R^\mu{}_\beta$$

3.2 The Christoffel Symbols and the Metric

Taking a derivative is "moving" something in a space, or spacetime. If I move in the $\hat{x}$ direction, I think "$\frac{\partial}{\partial x}$"! This is great for flat spaces, but as the above discussion suggests, moving in *curved* spaces (or spacetimes) means we have to think **covariant derivative**. So – what happens if we take the covariant derivative of the metric itself? (Whoa!)

The answer is: NOTHING! Since the metric defines the intrinsic geometry, it shouldn't change as we move from place to place. A curved spacetime has the same curvature properties everywhere. So, that means

$$D_\alpha g_{\mu\nu} = 0 \implies \text{Metric is the same everywhere!}$$

Writing this out in terms of Christoffel symbols, we get

$$\partial_\alpha g_{\mu\nu} - \Gamma^\beta{}_{\alpha\mu} g_{\beta\nu} - \Gamma^\beta{}_{\alpha\nu} g_{\mu\beta} = 0$$

because we need a negative sign when we take the covariant derivative of things with covariant indices. We can rewrite this as

$$\partial_\alpha g_{\mu\nu} = \Gamma^\beta{}_{\alpha\mu} g_{\beta\nu} + \Gamma^\beta{}_{\alpha\nu} g_{\mu\beta} \quad \text{Ⓐ}$$

If we cycle through the indices as we did before (*i.e.* cyclically swap α, μ, and ν) we get

$$\partial_\nu g_{\alpha\mu} = \Gamma^\beta{}_{\nu\alpha} g_{\beta\mu} + \Gamma^\beta{}_{\nu\mu} g_{\alpha\beta} \quad \text{Ⓑ}$$

$$\partial_\mu g_{\nu\alpha} = \Gamma^\beta{}_{\mu\nu} g_{\beta\alpha} + \Gamma^\beta{}_{\mu\alpha} g_{\nu\beta} \quad \text{Ⓒ}$$

I've called these Ⓐ, Ⓑ, Ⓒ. As before, we note that because of the symmetry of the metric under exchanges of indices, terms with $g_{\alpha\mu}$ are the same as those with $g_{\mu\alpha}$, and so forth. We can therefore whittle this expression down to solve for the Christoffel symbol by writing:

$$\text{Ⓒ} + \text{Ⓑ} - \text{Ⓐ}$$

$$\implies \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} = 2\Gamma^\beta{}_{\mu\nu}\, g_{\alpha\beta}$$

So, multiplying by the inverse of the metric on the RHS $g^{\alpha\beta}$, we obtain the same expression as before:

$$\Gamma^\beta{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right)$$

But this time we derived it by understanding that **the metric and geometry doesn't change when we move around through spacetime**.

3.3 The Geodesic Equation: Straight Lines in Curved Spacetime

First, what defines a straight line? From a Newtonian perspective, it's the **path of a free particle**. How do we describe a free particle in relativity? Simple: the same we do classically. **A free particle is one whose acceleration is zero**:

$$\frac{d\mathbf{u}}{d\tau} = 0 \quad , \quad \mathbf{u} = u^\mu \hat{e}_\mu$$

where the explicit dependence on the basis vector is included. Expanding this out, we get

$$\frac{d\mathbf{u}}{d\tau} = \frac{du^\mu}{d\tau}\hat{e}_\mu + u^\mu \frac{d\hat{e}_\mu}{d\tau} = 0$$

Furthermore, since $u^\mu = \frac{dx^\mu}{d\tau}$, this becomes

$$\frac{d^2x^\mu}{d\tau^2}\hat{e}_\mu + \frac{dx^\mu}{d\tau}\frac{d\hat{e}_\mu}{d\tau} = 0$$

What to do with the derivative of $\hat{e}_\mu$? Why, did you say "Christoffel symbol"? Great idea! Except, the Christoffel symbol is defined in terms of the coordinate derivative (∂_μ), and not $\frac{d}{d\tau}$. But never fear – as calculus has taught us, we know that we can address this implicit dependence as follows:

$$\frac{d\hat{e}_\mu}{d\tau} = \frac{\partial \hat{e}_\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}$$

and since

$$\frac{\partial \hat{e}_\mu}{\partial x^\nu} = \Gamma^\alpha{}_{\mu\nu}\hat{e}_\alpha \quad \text{(Note : I flipped } \mu \text{ and } \nu \text{ by symmetry)}$$

we get

$$\frac{\partial \hat{e}_\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau} = \Gamma^\alpha{}_{\mu\nu}\hat{e}_\alpha \frac{dx^\nu}{d\tau}$$

The entire expression is then

$$\frac{d^2x^\mu}{d\tau^2}\hat{e}_\mu + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Gamma^\alpha{}_{\mu\nu}\hat{e}_\alpha$$

Again, μ and α are paired indices here, so we can substitute one for the other. We'll replace the first one in the double derivative ($\mu \to \alpha$), since we don't want too many other μs in the second term:

$$0 = \frac{d^2x^\alpha}{d\tau^2}\hat{e}_\alpha + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Gamma^\alpha{}_{\mu\nu}\hat{e}_\alpha$$

and factoring out the $\hat{e}_\alpha$ term gives us

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{11}$$

This is called the **geodesic equation**, and is the equation that describes **a straight path (or a free particle) in curved spacetime**. Note that in *flat* spacetime, we would just have the equation $\frac{d^2x^\mu}{d\tau^2} = 0$, which tells us that **the Christoffel Symbols are zero in flat spacetime**.

3.4 The Riemann Curvature Tensor

The next application of the Christoffel symbols is the important one. Like, **the** important one! As I showed you earlier in the class, moving a vector around a closed path on a spherical surface rotates the direction it points. Furthermore, rotating it half-way around the closed path along two different routes rotates it by different amounts. How this vector changes are we move it along a particular direction can be mathematically quantified by considering derivatives.

Specifically, since we're asking the question "How does moving it along $\hat{\theta}$ first and $\hat{\phi}$ second compare to moving it along $\hat{\phi}$ first and $\hat{\theta}$ second?" In flat space, these two conditions would translate to:

$$\vec{A} \text{ along } \hat{\theta} \text{ first, } \hat{\phi} \text{ second} \quad \Longrightarrow \quad \frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta}\tilde{\mathrm{A}} = \frac{\partial^2}{\partial\phi\,\partial\theta}\tilde{\mathrm{A}}$$

$$\vec{A} \text{ along } \hat{\phi} \text{ first, } \hat{\theta} \text{ second} \quad \Longrightarrow \quad \frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi}\tilde{\mathrm{A}} = \frac{\partial^2}{\partial\theta\,\partial\phi}\tilde{\mathrm{A}}$$

As calculus has taught us, though, these two things are *equal*, because of the convenient fact of **equality of mixed partials**. So, each would give the same answer – both paths are the same... but only in flat space!

Of course, the situation is different when we consider curved spaces – but the *premise* is the same. To calculate how the two paths differ, and hence determine how they affect the vector, we consider the **covariant derivative along each path**. We'll change the notation a bit, so that instead of "$\hat{\theta}$" and "$\hat{\phi}$" (those specifically refer to angular coordinates), we'll talk $\hat{e}_\mu$ and $\hat{e}_\nu$. In this case:

$$A^\alpha \text{ along } \hat{\mathrm{e}}_\mu \text{ first, } \hat{\mathrm{e}}_\nu \text{ second} \quad \Longrightarrow \quad \mathrm{D}_\nu\mathrm{D}_\mu\mathrm{A}^\alpha$$

$$A^\alpha \text{ along } \hat{\mathrm{e}}_\nu \text{ first, } \hat{\mathrm{e}}_\mu \text{ second} \quad \Longrightarrow \quad \mathrm{D}_\mu\mathrm{D}_\nu\mathrm{A}^\alpha$$

But this time, these derivatives aren't equal! In fact, how they differ is explicitly related to the background curvature:

$$D_\mu D_\nu A^\alpha - D_\nu D_\mu A^\alpha = \text{ Something related to } A^\alpha \text{ with three free indices!}$$

Taking inspiration from how we defined the Christoffel symbols, we say that this "thing" relating A^α with three free indices is **the curvature tensor**:

$$\implies \boxed{D_\mu D_\nu A^\alpha - D_\nu D_\mu A^\alpha = R^\alpha{}_{\beta\mu\nu} A^\beta}$$

and since it needs a proper name, we'll call it.... uh... the **Riemann curvature tensor** (because I'm sure that, like with me, it's the first name that popped into your head).

Again, this result is some variety of linear combination of the components of the vector A. You can verify for yourself that it is consistent, from the index perspective.

How do we calculate the Riemann curvature tensor? Write it out! Remember, from before, the covariant derivative of a vector is $D_\mu A^\alpha = \partial_\mu A^\alpha + \Gamma^\alpha{}_{\mu\lambda} A^\lambda$, and for a co-vector it's $D_\mu A_\alpha = \partial_\mu A_\alpha - \Gamma^\lambda{}_{\mu\alpha} A_\lambda$. This holds for the partial derivative, too! In fact, we get

$$D_\mu \partial_\nu = \partial_\mu \partial_\nu - \Gamma^\lambda{}_{\mu\nu} \partial_\lambda$$

$$\begin{aligned} &\text{Term 1} \\ D_\mu D_\nu A^\alpha &= D_\mu \left(\partial_\nu A^\alpha + \Gamma^\alpha{}_{\nu\lambda} A^\lambda \right) \\ &= \partial_\mu \partial_\nu A^\alpha + \partial_\mu \Gamma^\alpha{}_{\nu\sigma} A^\sigma + \Gamma^\alpha{}_{\nu\sigma} \partial_\mu A^\sigma + \Gamma^\alpha{}_{\mu\sigma} \partial_\nu A^\sigma + \Gamma^\alpha{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\beta} A^\beta \\ &\quad - \Gamma^\sigma{}_{\mu\nu} \partial_\sigma A^\alpha - \Gamma^\sigma{}_{\mu\nu} \Gamma^\alpha{}_{\sigma\beta} A^\beta \end{aligned}$$

By defining Term 2 in a similar fashion, we can show that the difference results in something that looks like this:

$$D_\mu D_\nu A^\alpha - D_\nu D_\mu A^\alpha = R^\alpha{}_{\beta\mu\nu} A^\beta$$

as we saw above, where the Riemann curvature tensor is

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\mu\sigma} \Gamma^\sigma{}_{\beta\nu} - \Gamma^\alpha{}_{\nu\sigma} \Gamma^\sigma{}_{\beta\mu} \tag{12}$$

The biggest thing to note about this tensor is that it depends on **first and second derivatives of the metric only!** Aha! "General relativity is a rank-2 tensor theory of up to second order derivatives of the metric!" Higher-order derivatives don't figure into any condition in which we consider curvature. This will be important for the Einstein equations.

We can also define this in terms of what's called **geodesic deviation**, which is essentially a measure of whether or not parallel lines will ever meet! In flat spacetime, they don't (curvature is zero). In spaces of positive curvature, they converge! And in spaces of negative curvature, they diverge!