**PHYS 206 – Spring 2019**
**Assignment #5 solutions**

1. If we supose that the general force law is $F = ke^2 r^\beta$ for abitrary exponent $\beta$, then the Bohr quantization conditions can be derived as follows:

$$\text{Force balancing} \implies \frac{mv^2}{r} = ke^2 r^\beta$$
$$\text{Energy conservation} \implies E = \frac{1}{2}mv^2 + V(r)$$

In this case, the potential $V(r)$ is *not* the usual Coulomb ($\frac{1}{r}$) potential, since we need to remember that $F(r) = -\nabla V(r)$. Thus, we need to substitute $V(r) = \frac{ke^2}{\beta+1} r^{\beta+1}$. Noting also from the "force" condition that $mv^2 = ke^2 r^{\beta+1}$, the energy condition becomes

$$E = \frac{1}{2}ke^2 r^{\beta+1} + \frac{ke^2}{\beta+1} r^{\beta+1} = ke^2 r^{\beta+1}\left(\frac{1}{2} + \frac{1}{\beta+1}\right)$$

We can eliminate velocity from the conservation of angular momentum condition,

$$mvr = n\hbar \quad \longrightarrow \quad v^2 = \frac{n^2\hbar^2}{m^2 r^2}$$

and using the force condition, we can solve for $r$:

$$\frac{mv^2}{r} = ke^2 r^\beta$$
$$r^{\beta+1} = \frac{mv^2}{ke^2} = \frac{mn^2\hbar^2}{m^2 r^2 ke^2}$$
$$r^{\beta+3} = \frac{n^2\hbar^2}{mke^2} \implies r = \left(\frac{n^2\hbar^2}{mke^2}\right)^{\frac{1}{\beta+3}}$$

Substituting this back into the energy condition gives:

$$\begin{aligned} E &= k\left(\frac{n^2\hbar^2}{mke^2}\right)^{\frac{\beta+1}{\beta+3}}\left(\frac{1}{2} + \frac{1}{\beta+1}\right) \\ &= k(k)^{-\frac{\beta+1}{\beta+3}}\left(\frac{n^2\hbar^2}{me^2}\right)^{\frac{\beta+1}{\beta+3}}\left(\frac{1}{2} + \frac{1}{\beta+1}\right) \end{aligned}$$

Since $k(k)^{-\frac{\beta+1}{\beta+3}} = k^{\frac{\beta+3-\beta-1}{\beta+3}} = k^{\frac{2}{\beta+3}}$, we find that the modified Bohr energies are

$$\boxed{E = k^{\frac{2}{\beta+3}}\left(\frac{n^2\hbar^2}{me^2}\right)^{\frac{\beta+1}{\beta+3}}\left(\frac{1}{2} + \frac{1}{\beta+1}\right)}$$

2. (a) Since the atoms in the lattice are spaced at $a = 2.15\ \mathring{A}$, we can take this to be the "slit" spacing. The peak electron signal would result at the first maximum of the multiple-slit interference pattern, which can be determined from the expression $a \sin\theta = \lambda_{dB}$, where $\lambda_{dB}$ is the deBroglie wavelength of the electrons. Thus, we find $\boxed{\lambda_{dB} = 1.64\ AA = 1.64 \times 10^{-10}\ \text{m.}}$

(b) When an electron is passed through a potential difference $\Delta V$, it acquires an amount of energy $E = |q_e|\Delta V$, which by conservation of energy must be its kinetic energy. Note that the momentum of a particle with kinetic energy $K$ is $p = mv = \sqrt{2mK}$, so we can find the deBroglie wavelength as follows:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2m_e K}} = \frac{h}{\sqrt{2m_e|q_e|\Delta V}} = 1.67 \times 10^{-10}\ \text{m}$$

So, the two wavelengths agree! Louis deBroglie gets his Nobel Prize!

3. (a) The de Bröglie wavelength for a $m = 75$ kg person running at $v = 10$ m/s is $\boxed{\lambda_{dB} = \frac{h}{mv} = \frac{6.63\times10^{-34}}{(75)(10)} = 8.84 \times 10^{-37}\ \text{m}}$. The Compton wavelength is $\boxed{\lambda_C = \frac{h}{mc} = \frac{6.63\times10^{-34}}{(75)(3\times10^{8})} = 2.95 \times 10^{-44}\ \text{m}}$. The Compton wavelength is clearly much smaller than the de Bröglie wavelength.

(a) An electron traveling at $v = 0.3$ is somewhat relativistic, so we should include the relativistic momentum in the calculation of the de Bröglie wavelength. The gamma factor for $v = 0.3$ is $\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-(0.3)^2}} = 1.05$. So, we really don't need to include relativity, but since we already did the calculation, we'll proceed! The de Bröglie wavelength is $\boxed{\lambda_{dB} = \frac{h}{\gamma m_e v} = \frac{6.63\times10^{-34}}{(1.05)(9.1\times10^{-31})(0.9\times10^{8})} = 7.71 \times 10^{-12}\ \text{m}}$. This is slightly smaller than the atomic scale ($10^{-10}$ m), but larger than the nuclear scale ($10^{-15}$ m). The Compton wavelength, meanwhile, is $\boxed{\lambda_C = \frac{h}{m_e c} = \frac{6.63\times10^{-34}}{(9.1\times10^{-31})(3\times10^{8})} = 2.42 \times 10^{-12}\ \text{m}}$. We see that this is still in the same ballpark as the de Bröglie wavelength!

4.

$$c = 2.99 \times 10^{8}\ \frac{\text{m}}{\text{s}} \quad ; \quad \hbar = 1.05 \times 10^{-34}\ \frac{\text{kg}\cdot\text{m}^2}{\text{s}} \quad ; \quad G = 6.67 \times 10^{-11}\ \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$$

Using these fundamental constants, we can form quantities having specific units by raising each to a particular power: $\hbar^\alpha c^\beta G^\omega$, which will give a unit equation of the form

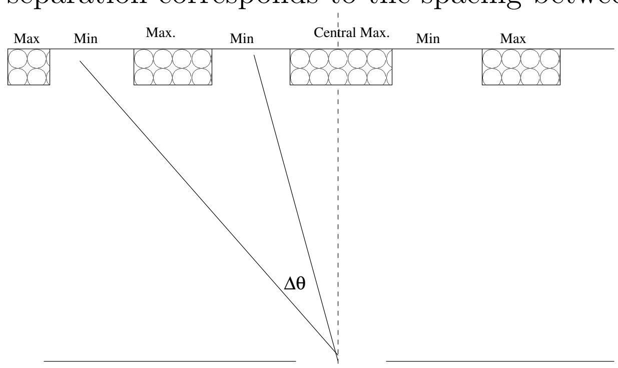
$$\left(\frac{\text{kg}\cdot\text{m}^2}{\text{s}}\right)^\alpha \left(\frac{\text{m}}{\text{s}}\right)^\beta \left(\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}\right)^\omega = (kg)^{\alpha-\omega}(m)^{2\alpha+\beta+3\omega}(s)^{-(\alpha+\beta+2\omega)}$$

(a) The Planck mass is derived from the system of exponent equations: $\alpha - \omega = 1, 2\alpha + \beta + 3\omega = 0, \alpha + \beta + 2\omega = 0$, which has constraints $\alpha = -\omega$. From this, one can show that $\beta = \frac{1}{2} = -\omega$. Thus, the Planck mass is $\boxed{M_{\text{Pl}} = \sqrt{\frac{\hbar c}{G}} = 2.17 \times 10^{-8} \text{ kg}}$

(b) To obtain units of time, we require: $\alpha - \omega = 0$, $2\alpha + \beta + 3\omega = 0$, $\alpha + \beta + 2\omega = -1$. The first equation implies that $\alpha = \omega$, and substituting into the second gives $\beta = -5\alpha$. The third equation yields $\alpha = -\frac{1}{2}$. Thus, the Planck time is $\boxed{T_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^5}} = 5.41 \times 10^{-44} \text{ s}}$

(c) By the same procedure, we can isolate units of length through the equations $\alpha - \omega = 0, 2\alpha + \beta + 3\omega = 1, \alpha + \beta + 2\omega = 0$, giving $\alpha = \omega = \frac{1}{2}$, $\beta = -\frac{3}{2}$. Thus, the Planck length is $\boxed{L_{\text{Pl}} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35} \text{ m}}$

5. For people to diffract in a single-slit experiment, we can assume that their separation corresponds to the spacing between each successive minimum:




For small angles, we have $\theta_n \approx \frac{n\lambda}{D}$, so between two successive angles $\theta_n$ and $\theta_{n+1}$, we find

$$\Delta\theta = \theta_{n+1} - \theta_n = \frac{(n+1)\lambda}{D} - \frac{n\lambda}{D} = \frac{\lambda}{D}$$

If the people are 1 m apart on the wall (5 m from the door), the angular separation is $\Delta\theta = \frac{1}{5} = 0.2$ rad. So, we have

$$\Delta\theta = \frac{h}{mvD}$$

and we can solve for Planck's Constant accordingly:
$\boxed{h = mvD\Delta\theta = 9.0 \text{ J·s.}}$ That is, Planck's Constant would have to be about 34 orders of magnitude bigger than it is for quantum effects to become apparent on such a large scale.